Double-Negation Elimination in Some Propositional Logics

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Abstract

This article answers two questions (posed in the literature), each concerning the guaranteed existence of proofs *free of double negation*. A proof is free of double negation if none of its deduced steps contains a term of the form $n(n(t))$ for some term t, where n denotes negation. The first question asks for conditions on the hypotheses that, if satisfied, guarantee the existence of a double-negation-free proof when the conclusion is free of double negation. The second question asks about the existence of an axiom system for classical propositional calculus whose use, for theorems with a conclusion free of double negation, guarantees the existence of a double-negation-free proof. After giving conditions that answer the first question, we answer the second question by focusing on the Łukasiewicz three-axiom system. We then extend our studies to infinite-valued sentential calculus and to intuitionistic logic and generalize the notion of being double-negation free. The double-negation proofs of interest rely exclusively on the inference rule condensed detachment, a rule that combines modus ponens with an appropriately general rule of substitution. The automated reasoning program Otter played an indispensable role in this study.

1 Origin of the Study

This article features the culmination of a study whose origin rests equally with two questions, the first posed in *Studia Logica* [2] and the second (closely related to the first) posed in the *Journal of Automated Reasoning* [15].

Both questions focus on *double-negation-free proofs*, proofs none of whose deduced steps contain a formula of the form $n(n(t))$ for some term t with the function n denoting negation. For example, where i denotes implication, the presence of the formula $i(i(n(x), x), x)$ as a deduced step does not preclude a proof from being double-negation free, whereas the presence of the formula $i(n(n(x)), x)$ does. Note the distinction between deduced steps and axioms; in particular, use of the Frege system for two-valued sentential calculus, which contains two axioms in which double negation occurs, guarantees the existence of double-negation-free proofs, as we show in Section 5.

The sought-after double-negation-free proofs of interest here rely solely on the inference rule condensed detachment [10], a rule that combines modus ponens with an appropriately general rule of substitution. Formally, condensed detachment considers two formulas, $i(A, B)$ (the major premiss) and C (the minor premiss), that are tacitly assumed to have no variables in common and, if C unifies with A, yields the formula D, where D is obtained by applying to B a most general unifier of C and A.

In [2], the following question is asked. Where P and Q may each be collections of formulas, if T is a theorem asserting the deducibility of Q from P such that Q is free of double negation, what conditions guarantee that there exists a proof of T (relying solely on condensed detachment) all of whose deduced steps are free of double negation? Then, in [15], Dolph Ulrich asks about the existence of an axiom system for two-valued sentential (or classical propositional) calculus such that, for each double-negation-free formula Q provable from the axiom system, there exists a double-negation-free proof of Q.

Although it is perhaps not obvious, the nature of the axioms chosen for the study of some area of logic or mathematics can have a marked impact on the nature of the proofs derived from them. As a most enlightening illustration of this relation and indeed pertinent to the two cited questions (each of which we answer in this article), we turn to an example given by Ulrich that builds on a result of C. A. Meredith. In the early 1950s, Meredith found the following 21-letter single axiom for two-valued logic.

$$i(i(i(i(i(x, y), i(n(z), n(u))), z), v), i(i(v, x), i(u, x)))$$

Consider the following system with condensed detachment as the sole rule of inference and four double-negation-free classical theses (of two-valued logic) as axioms. (The notation here is taken from Ulrich [15] and should *not* be confused with that used for infinite-valued logic discussed in Section 7.)

A1 $$i(x,x)$$

A2 $$i(i(x,x),i(n(x),i(n(x),n(x))))$$

A3 $$i(i(x,i(x,x)),i(n(x),i(n(x),i(n(x),n(x)))))$$

A4 $$i(i(x,i(x,i(x,x))),i(i(i(i(y,z),i(n(u),n(v))),u),w),i(i(w,y),i(v,y))))$$

One can readily verify that axiom A1 and the antecedent (left-hand argument) of A2 are unifiable but that no other axiom is unifiable with the antecedent of any axiom. In other words, no conclusion can be drawn (with condensed detachment) other than by considering A1 and A2. Therefore, the first step of any proof in this system can only be

5 $\quad i(n(x),i(n(x),n(x))).$

Similarly, the only new path of reasoning now available is that of 5 with the antecedent of A3. Therefore, the next step in any proof in this system can only be

6 $\quad i(n(n(x)),i(n(n(x)),i(n(n(x)),n(n(x))))).$

Of course, 6 and the antecedent of A4 are unifiable, and we may obtain

7 $\quad i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x))).$

But, since 7 is Meredith's single axiom for two-valued sentential calculus, we may then deduce all other theorems of classical sentential logic. The given four-axiom system does, therefore, provide a complete axiomatization for classical two-valued logic; but no proof of any classical theses except A1–A4 and 5 can be given that does not include at least formula 6, in which $n(n(x))$ (double negation) appears four times.

Thus one sees that some axiom systems have so much control over proofs derived from them that double negation is inescapable. As for the Meredith single axiom (derived from the Ulrich example), what is its status with regard to guaranteed double-negation-free proofs of theorems that themselves are free of double negation? Of a sharply different flavor, what is the status in this regard of the Frege axiom system in view of the fact that two of its members each contain a double negation, $i(n(n(x)),x)$ and $i(x,n(n(x)))$? The Frege axiom system consists of the following six axioms.

$$i(x,i(y,x))$$
$$i(x,n(n(x)))$$
$$i(n(n(x)),x)$$
$$i(i(x,i(y,z)),i(i(x,y),i(x,z)))$$
$$i(i(x,y),i(n(y),n(x)))$$
$$i(i(x,i(y,z)),i(y,i(x,z)))$$

These questions are also answered in this article as we complete our treatment of two-valued sentential calculus by giving conditions that, if satisfied by the axioms, guarantee the existence of a double-negation-free proof for each theorem that itself is double-negation free.

The study of this logical property of obviating the need for double negation demands its examination in other areas of logic and demands a natural extension. Therefore, we investigate this property in the context of infinite-valued sentential calculus and intuitionistic logic, and we present an extension of the property that focuses on theorems in which double negation appears.

Several of the results presented in this paper were found with the assistance of Otter [7], an automated reasoning program that searches for proofs. Otter has been used effectively to answer numerous open questions in a variety of algebras and logics. Its underlying logic is first-order predicate calculus with equality. Its inference rules are based on resolution [12]—a generalization of modus ponens and syllogism that includes instantiation—and equality substitution. Otter includes an extensive number of user-controlled strategies for directing the application of inference rules and for managing the potentially huge number of formulas (clauses) that can be deduced in a given study, and we rely heavily on these capabilities in our work.

Condensed detachment problems are easily represented for Otter. Axioms and theorems are represented as atomic formulas in a predicate `P` having the intuitive meaning "is a theorem". For example, the theorem $i(x,x)$ would be represented with the clause

```
P(i(x,x)),
```

where x is a universally quantified variable. The inference rule condensed detachment is effectively employed by including the clause

```
-P(i(x,y)) | -P(x) | P(y),
```

where "`-`" is the negation symbol and "`|`" represents disjunction, and by using the inference rule *hyperresolution* in Otter. The use of hyperresolution on clauses with symbol `P` corresponds to the use of condensed detachment on the formulas inside `P`. For the reader to whom this is new, we spell it out: Suppose $i(A, B)$ and C are two propositional formulas and that σ is a most general unifier of A and C. Then applying condensed detachment to $i(A, B)$ and C would deduce $B\sigma$. On the other hand, applying hyperresolution to the clauses `P(i(A,B))`, `P(C)` and the clause displayed above deduces the new clause `P(B`σ`)`.

In this study, Otter was used to prove conditions for double-negation elimination for a number of axiom systems. These conditions are instances of theorems that are not particularly difficult to prove with Otter. What makes these theorems interesting and challenging from an automated theorem-proving perspective is the requirement of finding derivations of the exact instances of the conditions. Resolution-based theorem provers, by design, draw and retain most general conclusions and can easily become overwhelmed if less general conclusions (proper instances) are retained. Finding derivations of the desired instances required careful application of—and in some cases modification to—the strategies that Otter typically relies on for the management of clauses.

Our interest in double-negation avoidance can be traced directly to our successes with Otter in previous work. In particular, a large number of proofs were obtained by applying a strategy that instructs Otter to avoid retention of any deduced conclusion if it contains a double-negation term. Use of this strategy sharply increased the likelihood of success. Because the literature strongly suggests that reliance on double negation is unavoidable, and because our completed proofs suggested the contrary, the questions that are central to this article were studied.

2 The Interplay of Axioms and Proof

Once posed, the question of double-negation avoidance seems quite natural, meshing well with other concerns for proof properties as expressed by logicians. For example, Meredith and Prior [6], then Thomas [14] avidly sought shorter and still shorter proofs; size of proof (total number of symbols) is of interest to Ulrich; and the dispensing with thought-to-be-key lemmas is almost always of general interest.

More familiar to many are similar concerns for the axioms of a theory. Indeed, in logic, merited emphasis is placed on the nature and properties of various axiom systems: the number of members, the length (individually and collectively), the number of distinct letters (variables), the total number of occurrences of various function symbols, and other measures of "simplicity". To mention but one of many examples, in the mid-1930s J. Łukasiewicz discovered a 23-letter single axiom for two-valued sentential (or classical propositional) calculus. As cited in Section 1, almost two decades later Meredith found a 21-letter single axiom. Whether a still shorter single axiom for this area of logic exists is currently unknown.

Although it is common to consider the properties of a proof or the properties of an axiom system, less work connects the two directly—for example, considering relationships of the form "if an axiom system has a property P_A, then every theorem necessarily has a proof satisfying property P_P". Here we study such a direct connection when we identify properties of an axiom system that guarantee the existence of a double-negation-free proof.

Double-negation-free proofs, in addition to their aesthetic appeal and their interest from a logical viewpoint, are relevant to the work of Hilbert. Indeed, although it was unknown until recently [20], Hilbert offered a twenty-fourth problem that was not included in the famous list of twenty-three seminal problems that he presented in Paris at the beginning of the twentieth century. This twenty-fourth problem focuses on the discovery of simpler proofs. Hilbert did not include the problem in his Paris talk, apparently because of the difficulty of defining "simpler" precisely.

Ceteris paribus, the avoidance of some type of term can make a proof simpler, as is the case when a proof is free of doubly negated subformulas. This paper, in the spirit of Hilbert's twenty-fourth problem, studies this specific form of simplicity, seeking (as noted) general sufficient conditions for an axiom system of propositional logic L that guarantees that doubly negated formulas that do not occur in the theorem are not needed in the proof.

3 Formalism

Although propositional calculus is one of the oldest areas of logic, not all of its mysteries have been unlocked. The existence of truth tables and other decision procedures for propositional logic notwithstanding, it is by no means trivial to prove, for example, that a given 23-symbol formula is in fact a single axiom. Truth tables and decision procedures can be used to determine whether a given formula is a tautology, and they may be helpful in constructing a proof of a given formula from a given set of axioms for propositional calculus, but generally they are not helpful in finding proofs of known axioms from other formulas (which is what one must do to verify that a formula is a single axiom). The search for such proofs has recently become a test bed in automated deduction. Not only do the theorems we prove here about double-negation elimination have an intrinsic, aesthetic appeal in that they show the possibility of simplifying proofs, but they also are of interest because they justify in the vast majority of cases a shortcut in automated proof-search methods, namely, the automatic discarding of double negations.

We shall work with logics formulated by using only the two connectives implication and negation. Several notations are in use for propositional logic that we mention before continuing. First, one can use infix $\rightarrow$ for implication and prefix $\neg$ for negation. For example, we could write $x \rightarrow (\neg x \rightarrow y)$. Closely related, many papers on propositional logic use Polish notation, in which **C** is used for implication (conditional) and **N** for negation. The same formula would then be rendered as $\mathbf{C}x\mathbf{CN}xy$. Finally, a notation that is appropriate when using Otter is prefix, with parentheses. We use $i(x, y)$ for implication and $n(x)$ for negation; therefore, the example formula would be $i(x, i(n(x), y))$. In this paper we use this last notation exclusively. It permits us to cut and paste machine-produced proofs, eliminating errors of transcription. We make use of the theorem-proving program Otter [7] to produce proofs in various propositional logics, proofs we use to verify that those logics satisfy the hypotheses of our general theorems on double-negation elimination.

Let L be Łukasiewicz's formulation of propositional calculus in terms of implication and negation, denoted by i and n, as given on page 221 of [19]. Łukasiewicz provided the following axiomatization of L.

$$\begin{array}{ll} \text{L1} & i(i(x, y), i(i(y, z), i(x, z))) \\ \text{L2} & i(i(n(x), x), x) \\ \text{L3} & i(x, i(n(x), y)) \end{array}$$

The inference rule frequently used in logic is known as condensed detachment. This rule (which is the only inference rule to be used in the sought-after double-negation-free proofs) combines substitution and modus ponens. Specifically, given a major premiss $i(p, q)$ and a minor premiss p, the conclusion of modus ponens is q. The substitution rule permits the deduction of $p\sigma$ from p, where σ is any substitution of terms for variables. Condensed detachment has premisses $i(p, q)$ and r and attempts to unify p and r—that is, seeks a substitution σ that makes $p\sigma = r\sigma$. If successful, provided σ is a most general such substitution, the conclusion of condensed detachment is $q\sigma$ or an alphabetic variant of $q\sigma$. This inference rule requires renaming of variables in the premisses before the attempted unification to avoid unintended clashes of variables.[1]

A double negation is a formula $n(n(t))$, where t is any term. A formula A *contains a double negation* if it has a not-necessarily-proper subformula that is a double negation. A derivation contains a double negation if one of its deduced formulas contains a double negation. Suppose that the formula A contains no double negations and is derivable in L. Then (central to this paper) does A have a derivation in L that contains no double negation? We answer this question in the affirmative (and thus answer the cited Ulrich question), not only for Łukasiewicz's system L1–L3, but also for other axiomatizations of classical (two-valued) propositional logic, as well as other systems of logic such as infinite-valued logic.

4 Condensed Detachment

We remind the reader that the systems of primary interest in this paper use condensed detachment as their *sole* rule of inference. For example, if α is a complicated formula and we wish to deduce $i(\alpha, \alpha)$, it would not be acceptable to first deduce $i(x, x)$ and then substitute α for x. Rather, it would be necessary to give a (longer) direct derivation of $i(\alpha, \alpha)$, relying solely on applications of condensed detachment.

We shall show in this section that our theorem about the eliminability of double negation holds for L1–L3 with condensed detachment if and only if it holds for L1–L3 with modus ponens and substitution. Similar results are in [3, 8], but for other systems: [3] treats the implicational fragment, while we allow negation, and [8] treats relevance logic. The following three formulas will play an important role.

[1] In the absence of the substitution rule, any alphabetic variant of an axiom is also accepted as an axiom. An "alphabetic" variant of A is a formula $A\sigma$, where the substitution σ is one-to-one and merely renames the variables. A technicality arises as to whether it is permitted, required, or forbidden to rename the variables of the premisses before applying condensed detachment. The definition on p. 212 of [19] does not explicitly mention renaming and, read literally, would not allow it, but the implementation in Otter requires it, and [15] explicitly permits it. If it is not permitted, then by renaming variables in the entire proof of the premiss, we obtain a proof of the renamed premiss, using alphabetic variants of the axioms, so the same formulas will be provable in either case. Similarly, renaming of variables in conclusions is allowed. Technically, we could wait until the conclusions are used before renaming them, but in practice, Otter renames variables in each conclusion as it is derived.

D1	$i(x,x)$
D2	$i(i(x,x),i(n(x),n(x)))$
D3	$i(i(x,x),i(i(y,y),i(i(x,y),i(x,y))))$

Lemma 1 *Suppose L is any system of propositional logic with condensed detachment as the sole inference rule, and suppose that there are proofs of D1–D3 in L. Then every formula of the form $i(\alpha,\alpha)$ is provable from L by condensed detachment. Furthermore, if there are double-negation-free proofs of D1–D3 in L, then $i(\alpha,\alpha)$ is provable without using double negations except those occurring as subformulas of α.*

Proof. We prove by induction on the complexity of the propositional formula α that for each α, the formula $i(\alpha,\alpha)$ is provable in L by condensed detachment. The base case, when α is a proposition letter, follows by replacing x by α in the proof of $i(x,x)$. Any line of the proof that is an axiom becomes an alphabetic variant of that axiom, which is still considered an axiom. Actually, in view of the convention that renaming variables in the conclusion is allowed, it would be enough just to replace x by α in the last line of the proof. If we have a proof of $i(\beta,\beta)$, then we can apply condensed detachment and D2 to get a proof of $i(n(\beta),n(\beta))$. This could introduce a double negation if β is already a negation, but in that case it is a double negation that already occurs in $\alpha = n(\beta)$, and so is allowed. Similarly, if we have proofs of $i(\alpha,\alpha)$ and $i(\beta,\beta)$, we can apply condensed detachment to D3 and get a proof of $i(i(\alpha,\beta),i(\alpha,\beta))$. That completes the proof of the lemma.

Lemma 2 *If A is an instance of C, then the result of applying condensed detachment to $i(A,B)$ and C is B (or an alphabetic variant of B).*

Proof: Rename variables in C if necessary so that C and A have no variables in common. Let σ be a most general unifier of A and C. Then the result of applying condensed detachment to $i(A,B)$ and C is $B\sigma$.

Let τ be a most general substitution such that $C\tau = A$; since A is assumed to be an instance of C, such a τ exists. Since the variables of C do not occur in A or B, $B\tau = B$ and $A\tau = A$. Then $C\tau = A = A\tau$, so $\tau = \sigma\rho$ for some substitution ρ. Then $B = B\tau = B\sigma\rho$. Thus $\sigma\rho$ is the identity on B. Hence $\sigma\rho$ is the identity on each variable occurring in B. Hence σ and ρ do nothing but (possibly) rename variables. Hence $B\sigma$, which is the result of this application of condensed detachment, is B or an alphabetic variant of B. That completes the proof of the lemma.

Lemma 3 *Suppose L is a logic proving D1–D3 by condensed detachment. Then each substitution instance α of an axiom of L is provable by condensed detachment. Furthermore, if L proves D1–D3 by condensed detachment without using double negations, then α is also provable without using double negations, except those double negations occurring as subformulas of α, if any.*

Proof. Let α be a substitution instance of an axiom A. Renaming the variables in the axiom A if necessary, we may assume that the variables occurring in A do not occur in α. By Lemma 1, $i(\alpha, \alpha)$ is provable by condensed detachment, without using any double negations except possibly those already occurring in α. By Lemma 2, the result of applying condensed detachment to $i(\alpha, \alpha)$ and A is α or an alphabetic variant $\alpha\sigma$ of α. If it is not literally α, we can rename variables in the conclusion (or, if one prefers to avoid renaming conclusions, throughout the entire proof) to create a proof of α. This completes the proof of the lemma.

A proof of B in L from assumptions Γ is defined as usual: Lines of the proof are inferred from preceding lines, or are axioms, or belong to Γ. When condensed detachment is used as a rule of inference, however, we have to distinguish between (propositional) variables that occur in the axioms and specific (constant) proposition letters that occur in assumptions. For example, if we have $i(n(n(x)), x)$ as an axiom, then we can derive any substitution instance of that formula, but if we have $i(n(n(a)), a)$ as an assumption, we cannot use it to derive an instance with some other formula substituted for a. We do not allow variables in assumptions, only constants.

The following theorem is the easy half of the relation between condensed-detachment proofs and modus ponens proofs. The sense of the theorem is that substitutions can be pushed back to the axioms.

Theorem 1 (Pushback theorem) *Let L be a system of propositional logic, and suppose L proves B by using condensed detachment or by using modus ponens and substitution. Then there exists a proof of B using modus ponens from substitution instances of axioms of L. Similarly, if L proves B from assumptions Δ, then there exists a proof of B using modus ponens from Δ and substitution instances of axioms of L.*

Remark. It would not make sense to speak of substitution instances of Δ because assumptions cannot contain variables, as explained above.

Proof. First we prove the theorem for the case when the given proof uses modus ponens and substitution. We proceed by induction on the length of the given proof of B. If the length is zero, then B is an axiom or assumption, and there is nothing to prove. If the last inference is by modus ponens, say B is inferred from $i(A, B)$ and A, then by the induction hypothesis there exist proofs of these premisses from substitution instances of axioms, and adjoining the last inference, we obtain the desired proof of B.

If the last inference is by substitution, say $B = A\sigma$ is inferred from A, then by the induction hypothesis there exists a proof π of A using modus ponens only from substitution instances of axioms. Apply the substitution σ to every line of π; the result is the desired proof of B. If there are assumptions, they are unaffected by σ because they do not contain variables.

Now suppose that the original proof uses condensed detachment. Each condensed-detachment inference can be broken into two substitutions and an application of modus ponens, so a condensed-detachment proof gives rise to a modus ponens and substitution proof, and we can apply the preceding part of the proof. That completes the proof.

The following lemma is not actually used in our work but is of independent interest. Condensed detachment is considered as an inference rule that combines modus ponens and substitution. The following lemma shows that it is reasonable to consider systems whose only rule of inference is condensed detachment, because such systems are already closed under the rule of substitution. This is not obvious *a priori* since condensed detachment permits only certain special substitutions.

Lemma 4 *Suppose L is a logic proving formulas D1–D3 by condensed detachment. If A is provable in L with condensed detachment and σ is any substitution, then $A\sigma$ is provable in L by condensed detachment.*

Proof. By induction on the length of the proof π of A in L, we prove that the statement of the lemma is true for all substitutions σ. The base case occurs when A is an axiom, so $A\sigma$ is a substitution instance of an axiom. By Lemma 3, $A\sigma$ is provable in L by condensed detachment.

For the induction step, suppose the last inference of the given proof π has premisses $i(p, q)$ and r, where τ is the most general unifier of p and r, and the conclusion is $q\tau = A$. By the induction hypothesis, we have condensed-detachment derivations of $i(p\tau\sigma, q\tau\sigma)$ and of $r\tau\sigma$. Since $p\tau = r\tau$, also $p\tau\sigma = r\tau\sigma$. Hence the inference from $i(p\tau\sigma, q\tau\sigma)$ and $r\tau\sigma$ to $q\tau\sigma$ is legal by condensed detachment. Hence we have a condensed-detachment proof of $q\tau\sigma = A\sigma$. This completes the proof of the lemma.

Theorem 2 (D-completeness) *Suppose L is a logic that proves formulas D1–D3. If L proves A by using modus ponens and substitution, then L proves A by using condensed detachment.*

Remark. We cannot track what happens to double negations in this proof. The proof does not guarantee that passing from substitution to condensed detachment will not introduce new double negations. Somewhat to our surprise, we do not need any such result to prove double-negation elimination; indeed, quite the reverse, we shall derive such a result from double-negation elimination.

Proof. By Theorem 1, there exists a proof π of A from substitution instances of axioms, using modus ponens as the only rule of inference. By Lemma 3, there exist condensed-detachment proofs of these substitution instances of axioms. Since modus ponens is a special case of condensed detachment, if we string together the condensed-detachment proofs of the instances of axioms required, followed by the proof π, we obtain a condensed-detachment proof of A. That completes the proof of the theorem.

5 The Main Theorem

Let L be a system of propositional logic, given by some axioms and the sole inference rule of condensed detachment. Let L* be the system of logic whose axioms are the closure of (the axioms of) L under applications of the following syntactic rule: If x is a proposition letter, and subterm $n(x)$ appears in a formula A, then construct a new formula by replacing each occurrence of x in A by $n(x)$ and cancelling any double negations that result. In other words, we choose a set S of proposition letters occurring negated in A, and we replace each occurrence of a variable x in S throughout A by $n(x)$, cancelling any doubly negated propositions. The first description of L* calls for replacing all occurrences of only *one* variable; but if we repeat that operation, we can in effect replace a subset.

An example will make the definition of L* clear. If this procedure is applied to the axiom

$$i(i(n(x), n(y)), i(y, x)),$$

we obtain the following three new axioms (by replacing first both x and y, then only y, then only x).

$$\begin{array}{ll} \text{A6} & i(i(x, y), i(n(y), n(x))) \\ \text{A7} & i(i(n(x), y), i(n(y), x)) \\ \text{A8} & i(i(x, n(y)), i(y, n(x))) \end{array}$$

We say that L admits double-negation elimination if, whenever L proves a theorem B, there exists a proof S of B in L such that any double negations occurring as subformulas in the deduced steps of S occur as subformulas of B.[2] In particular, *double-negation-free theorems have double-negation-free proofs* (ignoring the axioms).

Suppose B contains several doubly negated subformulas. We wish to consider eliminating double negations on just *some* of those subformulas. Let a subset of the doubly negated formulas in B be selected. Then let B^* be the result of erasing double negations on *all occurrences* of the selected subformulas in B. More precisely, B^* is obtained from B by replacing all occurrences of selected doubly negated subformulas $n(n(q))$ in B by q. We emphasize that if some doubly negated subformula occurs more than once in B, one must erase double negations on all or none of those occurrences. Generally there will be more than one way to select a set of doubly negated subformulas, so B^* is not unique. We say that L admits strong double-negation elimination if, whenever L proves a theorem B, and B^* is obtained from B as described, then there exists a proof of B^* in L, and moreover, there exists a proof of B^* in L that contains only doubly negated formulas occurring in B^*.

Theorem 3 *[Strong double-negation elimination]. Suppose that in L there exist double-negation-free proofs of D1–D3 and double-negation-free proofs of all the axioms of L*. Then L admits strong double-negation elimination.*

Remark. The theorem is also true with triple negation, quadruple negation, and so forth in place of double negation. For instance, if B contains a triple negation, then it has a proof containing no double negations not already contained in B. In particular, it then contains no triple negations not already contained in B, since every triple negation is a double negation.

Proof. Suppose B is provable in L. If B contains any double negations, select arbitrarily a subset of the doubly negated subformula of B, and form B^* by replacing each occurrence of these formulas $n(n(q))$ by q. Of course, B^* may still contain double negations; if we are proving only double-negation elimination and not strong double-negation elimination, we take B^* to be B. By Theorem 1, there is a modus ponens proof of B from substitution instances of axioms. If this proof contains any double negations that do not occur in B^*, we simply erase them. This erasure takes a modus ponens

[2]In this context, a formula t occurs as a subformula if and only if t or an alphabetic variant of t appears. For example, if $n(n(i(u,u)))$ occurs in B, then $n(n(i(x,x)))$ would be permitted in the deduced steps of S but not $n(n(i(x,y)))$ or $n(n(i(i(z,z),i(z,z))))$.

step into another legal modus ponens step. Note that one cannot "simply erase" double negations in a condensed-detachment proof; but now we have a modus ponens proof, and double negations *can* be erased in modus ponens proofs. For axioms, the process transforms a substitution instance of an axiom of L into a substitution instance of an axiom of L*. Thus we have a proof of B^* from substitution instances of axioms of L* that contains no double negations except those that already occur in B^*. By Lemma 3, there exist condensed-detachment proofs of these substitution instances of L* (from axioms of L*). By hypothesis, the axioms of L* have double-negation-free proofs in L. We now construct the desired proof as follows. First write the double-negation-free proofs of the axioms of L*. Then write proofs of the substitution instances of axioms of L* that are required. These actions provide proofs of all the substitution instances of axioms of L*, from L rather than from L*. Now write the proof of B^* from those substitution instances. We have the desired proof. The only double negations it contains are those contained in B^*. That completes the proof of the theorem.

Especially in view of the discussion focusing on the Frege axiom system, a natural question arises concerning its use as hypothesis. In particular, if the theorem to be proved is itself free of double negation, must there exist a double-negation-free proof of it with the Frege system as hypothesis? After all, that system contains two members exhibiting double negation. Because we have in hand a proof that deduces from the Frege system the featured Łukasiewicz axiom system such that the proof is free of double negation, such a proof must exist.

Theorem 4 (Strong D-completeness) *Suppose L is a logic that admits strong double-negation elimination. If L proves A using modus ponens and substitution, without using double negations except those that already occur as subformulas of A, then L proves A using condensed detachment, without using double negations except those that already occur as subformulas of A.*

Proof: Suppose L proves A using modus ponens and substitution. Then by Theorem 2, there is a condensed-detachment proof of A (possibly using new double negations). By strong double-negation elimination, there is a condensed-detachment proof of A in L, using only double negations that already occur as subformulas of A.

6 Łukasiewicz's System L1–L3

As mentioned in Section 3, Łukasiewicz's system L has the following axioms.

L1	$i(i(x,y),i(i(y,z),i(x,z)))$
L2	$i(i(n(x),x),x)$
L3	$i(x,i(n(x),y))$

Lemma 5 *From L1–L3, one can find double-negation-free proofs of formulas D1–D3.*

Proof. Formula D1 is $i(x,x)$. The following is a two-line proof produced by Otter.

1. $i(i(i(n(x),y),z),i(x,z))$ [L1, L3]
2. $i(x,x)$ [1, L2]

The following is an Otter proof of D2 from L1–L3.

1. $i(n(i(i(n(x),x),x)),y)$ [L3, L2]
2. $i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$ [L1, L1]
3. $i(i(x,y),i(n(i(i(n(z),z),z)),y))$ [L1, 1]
4. $i(i(i(n(x),y),z),i(x,z))$ [L1, L3]
5. $i(x,x)$ [4, L2]
6. $i(n(i(x,x)),y)$ [L3, 5]
7. $i(x,i(n(i(i(n(y),y),y)),z))$ [4, 3]
8. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [2, 2]
9. $i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))$ [8, L2]
10. $i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))$ [8, 9]
11. $i(i(x,i(n(y),y)),i(z,i(x,y)))$ [10, 7]
12. $i(i(n(x),y),i(z,i(i(y,x),x)))$ [2, 11]
13. $i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))$ [10, 12]
14. $i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))$ [8, 13]
15. $i(i(n(x),y),i(i(y,x),x))$ [13, 5]
16. $i(i(n(x),n(y)),i(y,x))$ [13, L3]
17. $i(x,i(y,x))$ [4, 16]
18. $i(i(n(x),y),i(x,x))$ [13, 17]
19. $i(i(x,x),i(x,x))$ [18, 6]
20. $i(i(x,i(y,z)),i(y,i(x,z)))$ [14, 17]

21.	$i(i(x,y),i(i(n(y),x),y))$	[20, 15]
22.	$i(n(x),i(x,y))$	[20, L3]
23.	$i(i(n(x),y),i(n(y),x))$	[13, 22]
24.	$i(n(i(x,n(y))),y)$	[23, 17]
25.	$i(i(x,i(n(y),z)),i(i(z,y),i(x,y)))$	[8, 21]
26.	$i(i(n(x),y),i(i(z,x),i(i(y,z),x)))$	[2, 25]
27.	$i(i(x,i(y,n(z))),i(i(z,x),i(y,n(z))))$	[26, 24]
28.	$i(i(x,y),i(n(y),n(x)))$	[27, L3]
29.	$i(i(i(n(x),n(y)),z),i(i(y,x),z))$	[L1, 28]
30.	$i(i(x,x),i(n(x),n(x)))$	[29, 19]

Finally, we are ready to prove D3. A proof of D3 was originally found using a specially compiled version of Otter. (The difficulty is that normal Otter derives a more general conclusion, which subsumes the desired conclusion.)

1.	$i(n(i(i(n(x),x),x)),y)$	[L3, L2]
2.	$i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$	[L1, L1]
3.	$i(i(x,y),i(n(i(i(n(z),z),z)),y))$	[L1, 1]
4.	$i(i(i(n(x),y),z),i(x,z))$	[L1, L3]
5.	$i(x,i(n(i(i(n(y),y),y)),z))$	[4, 3]
6.	$i(i(i(n(i(i(n(x),x),x)),y),z),i(u,z))$	[L1, 5]
7.	$i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$	[2, 2]
8.	$i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))$	[2, L1]
9.	$i(x,i(i(n(y),y),y))$	[6, L2]
10.	$i(i(x,i(n(y),y)),i(z,i(x,y)))$	[7, 9]
11.	$i(i(n(x),y),i(z,i(i(y,x),x)))$	[2, 10]
12.	$i(x,i(y,y))$	[10, L3]
13.	$i(x,i(i(n(y),z),i(i(z,y),y)))$	[10, 11]
14.	$i(i(n(x),y),i(i(y,x),x))$	[13, 13]
15.	$i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))$	[7, 14]
16.	$i(x,i(i(y,x),x))$	[4, 14]
17.	$i(i(x,i(y,z)),i(z,i(x,z)))$	[7, 16]
18.	$i(x,i(x,x))$	[17, 16]
19.	$i(x,i(y,x))$	[17, 9]
20.	$i(i(x,i(y,y)),i(x,i(y,y)))$	[18, 12]
21.	$i(i(x,i(y,x)),i(x,i(y,x)))$	[18, 19]
22.	$i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))$	[7, 15]
23.	$i(i(x,i(i(y,z),u)),i(i(y,v),i(x,i(i(v,z),u))))$	[7, 8]

24. $i(i(x,i(y,z)),i(y,i(x,z)))$ [22, 19]
25. $i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))$ [L1, 24]
26. $i(i(x,i(y,x)),i(y,i(x,x)))$ [25, 20]
27. $i(i(x,x),i(i(y,x),i(y,x)))$ [26, L1]
28. $i(i(x,i(i(y,z),u)),i(i(y,y),i(x,i(i(y,z),u))))$ [21, 23]
29. $i(i(x,x),i(i(y,y),i(i(x,y),i(x,y))))$ [28, 27]

That completes the proof of the lemma.

Theorem 5 *Łukasiewicz's system L1–L3 admits strong double-negation elimination.*

Proof. We begin by calculating the formulas L* for this system. We obtain the following.

$$\begin{array}{ll} \text{L4} & i(i(x,n(x)),n(x)) \\ \text{L5} & i(n(x),i(x,y)) \end{array}$$

By Theorem 3 it suffices to verify that there exist double-negation-free proofs of L4, L5, and D1–D3. We have already verified D1–D3 above, so it remains only to exhibit double-negation-free proofs of L4 and L5. The following is an Otter proof of L4.

1. $i(n(i(i(n(x),x),x)),y)$ [L3, L2]
2. $i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$ [L1, L1]
3. $i(i(x,y),i(n(i(i(n(z),z),z)),y))$ [L1, 1]
4. $i(i(i(n(x),y),z),i(x,z))$ [L1, L3]
5. $i(i(x,y),i(i(n(x),x),y))$ [L1, L2]
6. $i(x,x)$ [4, L2]
7. $i(x,i(n(i(i(n(y),y),y)),z))$ [4, 3]
8. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [2, 2]
9. $i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))$ [2, 5]
10. $i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))$ [8, 9]
11. $i(i(x,i(n(y),y)),i(z,i(x,y)))$ [10, 7]
12. $i(i(n(x),y),i(z,i(i(y,x),x)))$ [2, 11]
13. $i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))$ [10, 12]
14. $i(i(x,i(n(y),z)),i(i(u,i(z,y)),i(x,i(u,y))))$ [8, 13]
15. $i(i(n(x),n(y)),i(y,x))$ [13, L3]
16. $i(x,i(y,x))$ [4, 15]

17. $i(i(i(x,y),z),i(y,z))$ [L1, 16]
18. $i(n(x),i(x,y))$ [17, 15]
19. $i(i(n(x),y),i(n(y),x))$ [13, 18]
20. $i(i(x,i(y,z)),i(i(n(y),z),i(x,z)))$ [14, 19]
21. $i(i(n(x),y),i(i(x,y),y))$ [20, 6]
22. $i(i(x,n(x)),n(x))$ [21, 6]

The following is an Otter proof of L5.

1. $i(n(i(i(n(x),x),x)),y)$ [L3, L2]
2. $i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$ [L1, L1]
3. $i(i(x,y),i(n(i(i(n(z),z),z)),y))$ [L1, 1]
4. $i(i(i(n(x),y),z),i(x,z))$ [L1, L3]
5. $i(i(x,y),i(i(n(x),x),y))$ [L1, L2]
6. $i(x,i(n(i(i(n(y),y),y)),z))$ [4, 3]
7. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [2, 2]
8. $i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))$ [2, 5]
9. $i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))$ [7, 8]
10. $i(i(x,i(n(y),y)),i(z,i(x,y)))$ [9, 6]
11. $i(i(n(x),y),i(z,i(i(y,x),x)))$ [2, 10]
12. $i(i(x,i(y,z)),i(i(n(z),y),i(x,z)))$ [9, 11]
13. $i(i(n(x),n(y)),i(y,x))$ [12, L3]
14. $i(x,i(y,x))$ [4, 13]
15. $i(i(i(x,y),z),i(y,z))$ [L1, 14]
16. $i(n(x),i(x,y))$ [15, 13]

That completes the proof of the theorem.

Corollary 1 *Let T be any set of axioms for (two-valued) propositional logic. Suppose that there exist double-negation-free condensed-detachment proofs of L1–L3 from T. Then the preceding theorem is true with T in place of L1–L3.*

Proof. We must show that T admits strong double-negation elimination. Let A be provable from T, and let $A*$ be obtained from A by erasing some of the double negations in A (but all occurrences of any given formula if there are multiple occurrences of the same doubly negated subformula). We must show that T proves $A*$ by a proof whose doubly negated subformula occur in $A*$. Since T is an axiomatization of two-valued logic, $A*$ is a tautology and hence provable from L1–L3. By the theorem, there exists a proof of $A*$ from

L1–L3 that contains no double negations (except those occurring in $A*$, if any). Supplying the given proofs of L1–L3 from T, we construct a proof of $A*$ from T that contains no double negations except those occurring in $A*$ (if any). That completes the proof.

Example. We can take T to contain exactly one formula, the single axiom M of Meredith. M is double-negation free, and double-negation-free proofs of L1–L3 from M have been found using Otter [18]. Therefore, the theorem is true for the single axiom M.

7 Infinite-Valued Logic

Łukasiewicz's infinite-valued logic is a subsystem of classical propositional logic that was studied in the 1930s. The logic is of interest partly because there exists a natural semantics for it, according to which propositions are assigned truth values that are real (or rational) numbers between 0 and 1, with 1 being true and 0 being false. Łukasiewicz's axioms A1–A4 are complete for this semantics, as was proved (but apparently not published) by Wasjberg, and proved again by Chang [1]. Axioms A1–A4 are formulated by using implication $i(p, q)$ and negation $n(p)$ only. The truth value of p is denoted by $\|p\|$. Truth values are given by

$$\|n(p)\| = 1 - \|p\|$$

$$\|i(p, q)\| = \min(1 - \|p\| + \|q\|, 1).$$

Axioms A1–A4 are as follows.[3]

A1	$i(x, i(y, x))$
A2	$i(i(x, y), i(i(y, z), i(x, z)))$
A3	$i(i(i(x, y), y), i(i(y, x), x))$
A4	$i(i(n(x), n(y)), i(y, x))$

The standard reference for infinite-valued logic is [13].

Lemma 6 *A1–A4 prove formulas D1–D3 without double negation.*

Proof. The following is an Otter proof of D1 from A1–A4.

[3] In comparison with Łukasiewicz's axioms L1–L3, axiom A2 is the same as L1, and L3 is provable from A1–A4, but L2 is not provable from A1–A4.

1.	$i(x,i(y,i(z,y)))$	[A1, A1]
2.	$i(i(i(x,y),z),i(y,z))$	[A2, A1]
3.	$i(i(i(x,i(y,x)),z),z)$	[A3, 1]
4.	$i(x,x)$	[2, 3]

The following is an Otter proof of D2 from A1–A4, found by using a specially compiled version of Otter.

1.	$i(x,i(y,i(z,y)))$	[A1, A1]
2.	$i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$	[A2, A2]
3.	$i(i(i(x,y),z),i(y,z))$	[A2, A1]
4.	$i(n(x),i(x,y))$	[3, A4]
5.	$i(i(i(x,y),z),i(n(x),z))$	[A2, 4]
6.	$i(x,i(i(x,y),y))$	[3, A3]
7.	$i(i(i(i(x,y),y),z),i(i(i(y,x),x),z))$	[A2, A3]
8.	$i(i(i(x,i(y,x)),z),z)$	[A3, 1]
9.	$i(x,x)$	[3, 8]
10.	$i(i(i(x,x),y),y)$	[6, 9]
11.	$i(i(x,i(y,y)),i(y,y))$	[A3, 10]
12.	$i(i(x,x),i(x,x))$	[3, 11]
13.	$i(n(x),i(x,x))$	[5, 12]
14.	$i(i(x,i(y,x)),i(x,i(y,x)))$	[12, 1]
15.	$i(i(x,y),i(y,i(x,y)))$	[3, 14]
16.	$i(i(x,x),i(n(x),i(x,x)))$	[15, 13]
17.	$i(i(x,y),i(x,x))$	[2, 11]
18.	$i(x,i(i(y,z),i(y,y)))$	[A1, 17]
19.	$i(i(i(i(x,y),i(z,y)),i(z,y)),i(i(x,z),i(x,y)))$	[7, 2]
20.	$i(i(x,i(y,z)),i(x,i(y,y)))$	[19, 18]
21.	$i(i(x,x),i(n(x),n(x)))$	[20, 16]

The following is a proof of D3, found by using a specially compiled version of Otter.

1.	$i(x,i(y,i(z,y)))$	[A1, A1]
2.	$i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$	[A2, A2]
3.	$i(i(i(x,y),z),i(y,z))$	[A2, A1]
4.	$i(n(x),i(x,y))$	[3, A4]
5.	$i(x,i(y,i(z,x)))$	[3, A1]
6.	$i(x,i(i(x,y),y))$	[3, A3]

7. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [2, 2]
8. $i(i(x,y),i(z,i(x,z)))$ [2, 3]
9. $i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))$ [2, A2]
10. $i(x,i(n(y),x))$ [8, 4]
11. $i(n(x),i(y,i(z,i(u,z))))$ [10, 1]
12. $i(i(x,i(y,z)),i(y,i(x,z)))$ [7, 6]
13. $i(i(i(x,i(y,z)),u),i(i(y,i(x,z)),u))$ [A2, 12]
14. $i(x,i(y,y))$ [12, A1]
15. $i(x,i(y,i(z,i(u,u))))$ [5, 14]
16. $i(i(i(x,x),y),y)$ [A3, 14]
17. $i(i(x,i(y,y)),i(y,y))$ [A3, 16]
18. $i(i(x,i(y,y)),i(x,i(y,y)))$ [17, 15]
19. $i(i(x,i(y,x)),i(x,i(y,x)))$ [17, 11]
20. $i(i(x,i(i(y,z),u)),i(i(y,v),i(x,i(i(v,z),u))))$ [7, 9]
21. $i(i(x,i(y,x)),i(y,i(x,x)))$ [13, 18]
22. $i(i(x,x),i(i(y,x),i(y,x)))$ [21, A2]
23. $i(i(x,i(i(y,z),u)),i(i(y,y),i(x,i(i(y,z),u))))$ [19, 20]
24. $i(i(x,x),i(i(y,y),i(i(x,y),i(x,y))))$ [23, 22]

Theorem 6 *The system A1–A4 admits strong double-negation elimination.*

Proof. We begin by calculating the formulas L* for this system. The only axiom containing negations is A4, but there are three possible replacements, so we get three new axioms A6–A8 as follows.[4]

A6	$i(i(x,y),i(n(y),n(x)))$
A7	$i(i(n(x),y),i(n(y),x))$
A8	$i(i(x,n(y)),i(y,n(x)))$

By Theorem 3 it suffices to verify that there exist double-negation-free proofs of A6, A7, A8, and D1–D3. We have already verified D1–D3 above, so it remains only to produce double-negation-free proofs of A6–A8.

The following is an Otter proof of A6.

1. $i(x,i(y,i(z,y)))$ [A1, A1]
2. $i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$ [A2, A2]
3. $i(i(i(x,y),z),i(i(n(y),n(x)),z))$ [A2, A4]

[4]The name A5 is already in use for another formula, originally used as an axiom along with A1–A4, but later shown to be provable from A1–A4.

4. $i(i(i(x,y),z),i(y,z))$ [A2, A1]
5. $i(n(x),i(x,y))$ [4, A4]
6. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [2, 2]
7. $i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))$ [2, A2]
8. $i(i(x,y),i(n(y),i(x,z)))$ [6, 5]
9. $i(i(i(n(x),y),z),i(i(i(x,u),y),z))$ [7, 5]
10. $i(x,i(i(x,y),y))$ [4, A3]
11. $i(i(i(x,i(y,x)),z),z)$ [A3, 1]
12. $i(i(x,i(y,z)),i(y,i(x,z)))$ [6, 10]
13. $i(i(n(x),n(i(y,i(z,y)))),x)$ [3, 11]
14. $i(i(i(x,y),z),i(i(x,u),i(i(u,y),z)))$ [12, 7]
15. $i(i(x,y),i(i(z,x),i(z,y)))$ [12, A2]
16. $i(i(x,i(y,z)),i(i(z,u),i(x,i(y,u))))$ [6, 15]
17. $i(i(i(x,y),z),i(i(x,u),i(n(u),z)))$ [16, 8]
18. $i(i(n(x),y),i(n(y),x))$ [17, 13]
19. $i(i(i(x,y),z),i(n(z),x))$ [9, 18]
20. $i(i(i(x,y),z),i(i(z,u),i(n(u),x)))$ [14, 19]
21. $i(i(x,y),i(n(y),n(x)))$ [20, 13]

The following is an Otter proof of A7.

1. $i(x,i(y,i(z,y)))$ [A1, A1]
2. $i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$ [A2, A2]
3. $i(i(i(x,y),z),i(i(n(y),n(x)),z))$ [A2, A4]
4. $i(i(i(x,y),z),i(y,z))$ [A2, A1]
5. $i(n(x),i(x,y))$ [4, A4]
6. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [2, 2]
7. $i(i(x,y),i(n(y),i(x,z)))$ [6, 5]
8. $i(x,i(i(x,y),y))$ [4, A3]
9. $i(i(i(x,i(y,x)),z),z)$ [A3, 1]
10. $i(i(x,i(y,z)),i(y,i(x,z)))$ [6, 8]
11. $i(i(n(x),n(i(y,i(z,y)))),x)$ [3, 9]
12. $i(i(x,y),i(i(z,x),i(z,y)))$ [10, A2]
13. $i(i(x,i(y,z)),i(i(z,u),i(x,i(y,u))))$ [6, 12]
14. $i(i(i(x,y),z),i(i(x,u),i(n(u),z)))$ [13, 7]
15. $i(i(n(x),y),i(n(y),x))$ [14, 11]

The following is an Otter proof of A8.

1.	$i(x, i(y, i(z, y)))$	[A1, A1]
2.	$i(i(i(i(x, y), i(z, y)), u), i(i(z, x), u))$	[A2, A2]
3.	$i(i(i(x, y), z), i(i(n(y), n(x)), z))$	[A2, A4]
4.	$i(i(i(x, y), z), i(y, z))$	[A2, A1]
5.	$i(n(x), i(x, y))$	[4, A4]
6.	$i(i(x, i(y, z)), i(i(u, y), i(x, i(u, z))))$	[2, 2]
7.	$i(i(x, y), i(i(i(x, z), u), i(i(y, z), u)))$	[2, A2]
8.	$i(i(x, y), i(n(y), i(x, z)))$	[6, 5]
9.	$i(i(i(n(x), y), z), i(i(i(x, u), y), z))$	[7, 5]
10.	$i(x, i(i(x, y), y))$	[4, A3]
11.	$i(i(i(x, i(y, x)), z), z)$	[A3, 1]
12.	$i(i(x, i(y, z)), i(y, i(x, z)))$	[6, 10]
13.	$i(i(n(x), n(i(y, i(z, y)))), x)$	[3, 11]
14.	$i(i(i(x, y), z), i(i(x, u), i(i(u, y), z)))$	[12, 7]
15.	$i(i(x, y), i(i(z, x), i(z, y)))$	[12, A2]
16.	$i(i(x, i(y, z)), i(i(z, u), i(x, i(y, u))))$	[6, 15]
17.	$i(i(x, y), i(i(z, u), i(i(y, z), i(x, u))))$	[2, 16]
18.	$i(i(i(x, y), z), i(i(x, u), i(n(u), z)))$	[16, 8]
19.	$i(i(n(x), y), i(n(y), x))$	[18, 13]
20.	$i(i(i(x, y), z), i(n(z), x))$	[9, 19]
21.	$i(i(i(x, y), z), i(i(z, u), i(n(u), x)))$	[14, 20]
22.	$i(i(x, y), i(n(y), n(x)))$	[21, 13]
23.	$i(i(x, y), i(i(i(n(z), n(u)), x), i(i(u, z), y)))$	[17, 22]
24.	$i(i(i(n(x), n(y)), i(n(z), n(u))), i(i(y, x), i(u, z)))$	[23, A4]
25.	$i(i(x, n(y)), i(y, n(x)))$	[24, 19]

This completes the proof of the theorem.

8 An Intriguing Example

One of the motivations for this work was the existence of a formula that is double-negation free and provable from A1–A4 but for which Wos had been unable to find a double-negation-free proof. The formula in question is

$$\text{DN1} \qquad i(i(n(x), n(i(i(n(y), n(z)), n(z)))), n(i(i(n(i(n(x), y)), n(i(n(x), z))), n(i(n(x), z))))).$$

Wos provided a proof of 45 condensed-detachment steps of this theorem, 16 of whose lines involved a double negation. Beeson used this proof as input to a computer program implementing the algorithms implicit in the proof of our main theorem. The output of this program was a double-negation-free proof by modus ponens of the example, from substitution instances of A1–A4. The proof's length and size were surprising. It was 796 lines, and many of its lines involved thousands of symbols. The input proof takes about 3.5 kilobytes, the output proof about 200 kilobytes. Now we know what the *condensed* means in "condensed detachment"! The expansion in size is due to making the substitutions introduced by condensed detachment explicit. The expansion in length is due to duplications of multiply referenced lines, which must be done before the substitutions are "pushed upward" in the proof. In other words, one line of the proof can be referenced several times, and when the proof is converted to tree form, each reference will require a separate copy of the referenced line. This 796-line proof, considered as a tree, has substitution instances of the axioms at the leaves. After obtaining this proof, we could have continued with the algorithm, providing proofs of the substitution instances of the axioms. That approach would have substantially increased the length. Instead, McCune put the lines of the 796-line proof into an Otter input file as "hints" [16], and Otter produced a 27-line double-negation-free condensed-detachment proof of DN1 from A1–A4 and A6–A8. This run generates some 6,000 formulas and takes between one-half and two hours, depending on what machine is used. If the lines of this proof, together with the proofs of A6–A8, are supplied as resonators [17] in a new input file, Otter can then find a 37-step proof of DN1 from A1–A4.

9 D-Completeness of Intuitionistic Logic

Let H be the following formulation of intuitionistic propositional calculus in terms of implication and negation, denoted by i and n.[5]

H1	$i(x, i(y, x))$
H2	$i(i(x, i(y, z)), i(i(x, y), i(x, z)))$
H3	$i(i(x, n(x)), n(x))$
H4	$i(x, i(n(x), y))$

[5]These axioms can be found in Appendix I of [11], as the Łukasiewicz 2-basis in 12.1 plus the two axioms labeled (4) of 3.2, as specified in 12.5. According to [11], if we also add (2) and (3) of 3.2, we get the full intuitionistic propositional calculus; but (2) and (3) of 3.2 concern disjunction and conjunction. If they are omitted, the four axioms listed form a 4-basis for the implication-negation fragment. This will be proved in Corollary 2.

The inference rules of H are modus ponens and substitution. It is also possible to consider H1–H4 with condensed detachment. These two systems have the same theorems, as will be shown in detail below.

We note that H does not satisfy strong double negation elimination. Substituting $n(y)$ for x in axiom H3 produces $i(i(n(y), n(n(y))), n(n(y)))$. Cancelling the double negations produces $i(i(n(y), y), y)$, which is not provable in intuitionistic logic. This same example demonstrates directly that H does not satisfy the hypothesis of Theorem 3. Nevertheless, and perhaps surprising, H does satisfy double negation elimination—but we shall need a different proof to show that.

Lemma 7 *D1–D3 have double-negation-free condensed-detachment proofs from H1–H4.*

Proof: The following is a double-negation-free condensed-detachment proof of D1 from H1–H4.

1.	$i(i(x, y), i(x, x))$	[H2, H1]
2.	$i(x, x)$	[1, H1]

D2 is $i(i(x, x), i(n(x), n(x)))$. The following is a double-negation-free condensed-detachment proof of D2 from H1–H4, found by using a specially compiled version of Otter. Curiously, H3 is not used.

1.	$i(x, i(y, i(z, y)))$	[H1, H1]
2.	$i(x, i(y, i(z, i(u, z))))$	[H1, 1]
3.	$i(i(i(x, i(y, z)), i(x, y)), i(i(x, i(y, z)), i(x, z)))$	[H2, H2]
4.	$i(i(x, n(x)), i(x, y))$	[H2, H4]
5.	$i(i(x, y), i(x, x))$	[H2, H1]
6.	$i(x, i(i(y, z), i(y, y)))$	[H1, 5]
7.	$i(x, i(i(y, n(y)), i(y, z)))$	[H1, 4]
8.	$i(i(x, i(y, z)), i(x, i(y, y)))$	[H2, 6]
9.	$i(i(x, i(x, y)), i(x, y))$	[3, 5]
10.	$i(i(x, i(i(y, x), z)), i(x, z))$	[3, 1]
11.	$i(i(x, i(x, x)), i(x, x))$	[3, 9]
12.	$i(i(x, i(y, x)), i(x, i(y, x)))$	[11, 2]
13.	$i(n(x), i(x, y))$	[10, 7]
14.	$i(n(x), i(x, x))$	[8, 13]
15.	$i(i(x, y), i(y, i(x, y)))$	[12, 1]
16.	$i(i(x, x), i(n(x), i(x, x)))$	[15, 14]

17. $i(i(x,x),i(n(x),n(x)))$ [8, 16]

The following is a double-negation-free condensed-detachment proof of D3 from H1–H4. H3 and H4 are not used.

1. $i(x,i(y,i(z,y)))$ [H1, H1]
2. $i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))$ [H2, H2]
3. $i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))$ [H1, H2]
4. $i(i(x,y),i(x,x))$ [H2, H1]
5. $i(i(x,i(x,y)),i(x,y))$ [2, 4]
6. $i(i(x,i(i(y,x),z)),i(x,z))$ [2, 1]
7. $i(x,i(i(y,i(i(z,y),u)),i(y,u)))$ [H1, 6]
8. $i(i(x,y),i(i(z,x),i(z,y)))$ [6, 3]
9. $i(i(x,x),i(x,x))$ [5, 8]
10. $i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))$ [H2, 8]
11. $i(i(x,i(y,y)),i(x,i(y,y)))$ [8, 9]
12. $i(i(x,i(y,x)),i(x,i(y,x)))$ [9, 1]
13. $i(i(x,x),i(i(y,x),i(y,x)))$ [11, 8]
14. $i(i(i(x,y),z),i(y,z))$ [6, 7]
15. $i(i(x,y),i(i(y,z),i(x,z)))$ [14, 10]
16. $i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))$ [15, 15]
17. $i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))$ [16, 16]
18. $i(i(x,y),i(i(i(x,z),u),i(i(y,z),u)))$ [16, 15]
19. $i(i(x,i(i(y,z),u)),i(i(y,v),i(x,i(i(v,z),u))))$ [17, 18]
20. $i(i(x,i(i(y,z),u)),i(i(y,y),i(x,i(i(y,z),u))))$ [12, 19]
21. $i(i(x,x),i(i(y,y),i(i(x,y),i(x,y))))$ [20, 13]

Theorem 7 *The same theorems are provable from H1–H4 by using condensed detachment as the sole rule of inference as when we use modus ponens and substitution as rules of inference. Moreover, if b is provable without double negation by modus ponens from substitution instances of axioms, then there is a double-negation-free condensed-detachment proof of b.*

Remark. The present proof gives no assurance that a general H proof, using substitution arbitrarily and not just in axioms, can be converted to a condensed-detachment proof without introducing additional double negations. That stronger claim is also true: it is a consequence of Theorem 8 below.

Proof: The first claim is an immediate consequence of Theorem 2 and Lemma 7. To prove the second claim, suppose b has a double-negation-free modus ponens proof from substitution instances of axioms. By Lemma 3, we can supply double-negation-free condensed-detachment proofs of the substitution instances of axioms that are used in the proof. Adjoining these proofs, we obtain a double-negation-free condensed-detachment proof of b as required.

10 H and Sequent Calculus

Let G1 be the intuitionistic Gentzen calculus as given by Kleene [5]. Let G be G1 (minus cut), restricted to implication and negation; that is, formulas containing other connectives are not allowed. Thus the rules of inference of G are the four rules involving implication and negation, plus the structural rules. The rules of G1 are listed on pp. 442–443 of [5]. They will also be given in the course of the proof of Lemma 12. We shall use the notation $\Gamma \Rightarrow \Delta$ for a sequent. We remind the reader that what distinguishes intuitionistic from classical sequent calculus is that the consequent Δ in a sequent $\Gamma \Rightarrow \Delta$ in the intuitionistic calculus is restricted to contain at most one formula.[6]

We give a translation of H into G: If A is a formula of H, then A^0 is a formula of G, obtained by the following rules.

$$\begin{aligned} i(a,b)^0 &= a^0 \to b^0 \\ n(a)^0 &= \neg a^0 \end{aligned}$$

Of course, when a is a proposition letter (variable), then a^0 is just a. If $\Gamma = A_0, \ldots, A_n$ is a list of formulas of L, then Γ^0 is the list $A_0^0, \ldots, A_n^0$.

We translate G into H in the following manner. First we assign to each formula A of G a corresponding formula A' of H, given by

$$\begin{aligned} (A \to B)' &= i(A', B') \\ (\neg A)' &= n(A'), \end{aligned}$$

where again $A' = A$ for proposition letters A. We need to define Γ' also, where Γ is a list of formulas; since we are treating only the intuitionistic

[6] The translations given here can also be given for Łukasiewicz's logic L1–L3, but many additional complications are introduced by the necessity of translating a sequent containing more than one formula on the right, and in view of the simpler proofs of double-negation elimination given above, we treat the Gentzen translation only for intuitionistic logic. Note that we used Otter only for the H-proofs of D2 and D3; but if we treat L this way instead of H, we need Otter for twenty-one additional lemmas.

calculus, we need this definition only for lists occurring on the left of $\Rightarrow$. If $\Gamma = A_1, \ldots, A_n$ is a list of formulas occurring on the left of $\Rightarrow$, then Γ' is $A_1', \ldots, A_n'$.[7]

These two translations are inverse.

Lemma 8 *Let A be a formula of H. Then $A^{0'} = A$.*

Proof. By induction on the complexity of A. If A is a variable, then $A^0 = A$ and $A^{0'} = A$. We have

$$\begin{aligned} i(x,y)^{0'} &= (x^0 \to y^0)' \\ &= i(x^{0'}, y^{0'}) \\ &= i(x,y), \end{aligned}$$

and we have

$$\begin{aligned} n(x)^{0'} &= (\neg(x^0))' \\ &= n(x^{0'}) \\ &= n(x). \end{aligned}$$

Henceforth we simplify our notation by using lower-case letters for formulas of H and upper-case letters for formulas of G. Then we can write a instead of A' and A instead of a^0. By the preceding lemma, this convenient notation presents no ambiguity. Thus, for example, $(A \to B)'$ is $i(a,b)$. Greek letters are used for lists of formulas.

The following lemma gives several variations of the deduction theorem for H.

Lemma 9 (Deduction theorem for H) *(i) If H proves a from assumptions δ, b, then $i(b,a)$ is a theorem proved in H from assumptions δ, provided the assumptions contain only constant proposition letters.*

(ii) If a is provable from assumptions δ, b by condensed detachment from H1–H4, then $i(b,a)$ is derivable by condensed detachment from δ, provided the assumptions contain only constant proposition letters.

[7] A similar translation has been given in [9] in connection with Łukasiewicz's multivalued logics (which include the infinite-valued logic discussed in Section 7 of this paper). It is the obvious translation of Gentzen calculus into the implication-and-negation fragment of propositional calculus. We cannot appeal to any of the results of [9] because we are dealing with different logics, and besides we need to pay attention to double negations.

(iii) If there exists a proof of a by modus ponens from δ, b and substitution instances of H1–H4, then there exists a proof of $i(a, b)$ by modus ponens from δ and substitution instances of H1–H4.

(iv) In part (i), if the given proof of a has no double negations, then the proof of $i(b, a)$ from δ has no double negations.

(v) In part (iii), if the given proof of a has no double negations, then the proof of $i(b, a)$ from δ has no double negations.

Remarks: We do not prove a claim about double negations for condensed-detachment proofs, only for modus ponens proofs. That is, for condensed-detachment proofs, there is no part (vi) analogous to parts (iv) and (v). The reason for the restriction to constant assumptions in (i) and (ii) is the following. From the assumption $i(n(n(x)), x)$, we can derive any theorem of classical logic, for instance $i(n(n(a)), a)$, by substitution or condensed detachment. But we cannot derive the proposition that the first of these implies the second, $i(i(n(n(x)), x), i(n(n(a)), a))$. Therefore the deduction theorem is false without the restriction. Proofs by modus ponens from substitution instances of axioms do not suffer from this difficulty, which is one reason they are so technically useful in this paper.

Proof. First we show that (ii) follows from (iii). If we are given a condensed-detachment proof of a from assumptions δ, b using H1–H4, we can find, by Theorem 1, a modus ponens proof of a from δ, b and substitution instances of H1–H4. Applying (iii), we have a modus ponens proof of $i(b, a)$ from δ and substitution instances of H1–H4. By Theorem 7, this proof can be converted to a condensed-detachment proof of $i(b, a)$ from δ, completing the derivation of (ii) from (iii).

Next we show that (i) follows from (iii). Suppose we are given a proof of a from δ, b in H. By Theorem 1, we can find a modus ponens proof of a from assumptions δ, b and substitution instances of H1–H4. By (iii) we then can find a modus ponens proof of $i(b, a)$ from δ and substitution instances of H1–H4. Adding one substitution step above each such substitution instance, we have a proof in H of $i(b, a)$ from δ. That completes the proof that (i) follows from (iii).

We now show that (v) implies (iv). To do so requires going over the preceding paragraph with attention to double negations. Suppose we are given a double-negation-free proof of a from δ, b in H. By Theorem 1, we can find a modus ponens proof of a from assumptions δ, b and substitution instances of H1–H4, which is also double-negation-free. By (v) we then can find a double-negation-free modus ponens proof of $i(b, a)$ from δ and substitution instances of H1–H4. Adding one substitution step above each such substitution instance, we have a double-negation-free proof in H of $i(b, a)$ from δ. That completes the proof that (iv) follows from (v).

Now we prove (iii) and (v) simultaneously by induction on the number of steps in a pure modus ponens proof of a from δ and substitution instances of H1–H4.

Base case: a is b, or a member of δ, or a substitution instance of one of H1–H4.

If a is a substitution instance of an axiom of H1–H4, then by Lemma 3 there exists a condensed-detachment proof of a from H1–H4 that contains only double negations already occurring in a.

If a is b, then we use the fact that $i(b, b)$ is a theorem of H, provable without double negations (except those occurring in b) by Lemma 1. Hence by Theorem 1, it is provable by modus ponens from substitution instances of H1–H4.

If a is a member of δ, then consider the formula $i(a, i(b, a))$, which is a substitution instance of axiom H1. We can deduce $i(b, a)$ by modus ponens from this formula and a; adjoining this step to a one-step proof of a from δ "by assumption", we have a proof of $i(b, a)$ from δ.

Turning to the induction step, suppose the last step in the given proof infers a from $i(p, a)$ and p. By the induction hypothesis, we have proofs of $i(b, p)$ and $i(b, i(p, a))$ from δ. By axiom H2 and modus ponens (which is a special case of condensed detachment) we have $i(i(b, p), i(b, a))$. Applying modus ponens once more, we have $i(b, a)$ as desired. Note that no double negations are introduced. That completes the proof of the lemma.

We shall call a sequent $\Gamma \Rightarrow \Delta$ double-negation-free if it contains no double negation.

We shall refer to proofs by modus ponens from substitution instances of H1–H4 as M-proofs for short. M-proofs use modus ponens only but can use substitution instances of axioms, as opposed to H-proofs, which can use substitution anywhere as well as modus ponens. We have already shown how to convert condensed-detachment proofs to M-proofs (in Theorem 1), and vice versa (since every substitution instance of the axioms is derivable by condensed detachment).

Lemma 10 *If the final sequent $\Gamma \Rightarrow \Theta$ of a G-proof is double-negation free, then the entire G-proof is double-negation free.*

Proof. By the subformula property of cut-free proofs: Every formula in the proof is a subformula of the final sequent.

Lemma 11 *The translation from H to G is sound. That is, if H proves a from assumptions δ, then G proves the sequent $\Delta \Rightarrow A$ (where A is the translation a^0, and Δ is δ^0).*

Proof. We proceed by induction on the length of proofs. When the length is zero, we must exhibit a proof in G of each of the axioms H1–H4. This is a routine exercise in the Gentzen sequent calculus, which we omit. For the induction step, suppose we have proofs in H from assumptions δ of a and $i(a,b)$. Then by the induction hypothesis, we have proofs in G of $\Delta \Rightarrow A$ and $\Delta \Rightarrow A \to B$. We require a proof in G of $\Delta \Rightarrow B$. It is easy to exhibit a proof in G of $A \to ((A \to B) \to B)$. Applying the cut rule twice, we obtain a proof in G plus the cut rule of $\Delta \Rightarrow B$. By Gentzen's cut-elimination theorem, there exists a proof in G of $\Delta \Rightarrow B$. This completes the proof of the lemma.

Lemma 12 *(i) Suppose G proves the sequent $\Gamma \Rightarrow A$. Then there is an M-proof of a from assumptions γ. If G proves $\Gamma \Rightarrow []$, where $[]$ is the empty list, then there is an M-proof of p from assumptions γ, where p is any formula of H.*

(ii) If any double negations occur in subformulas of the given sequent $\Gamma \Rightarrow \Delta$ (where here Δ can be empty or not), then a proof as in (i) can be found that contains no double negations except those arising from the H-translations of double-negated subformulas of $\Gamma \Rightarrow \Delta$.

(iii) If in part (i) the H-translation of the given sequent $\Gamma \Rightarrow \Delta$ does not contain any double negations, then the M-proof that is asserted to exist can also be found without double negations.

Proof. We proceed by induction on the length of proof of $\Gamma \Rightarrow A$ in G.

Base case: the sequent has the form $\Gamma, A \Rightarrow A$. We must show that a is derivable in H from premisses γ, a, which is clear.

Now for the induction step. We consider one case for each rule of G.

Case 1, the last inference in the G-proof is by rule $\to\Rightarrow$:

$$\frac{\Delta \Rightarrow A \qquad B, \Gamma \Rightarrow \Theta}{A \to B, \Delta, \Gamma \Rightarrow \Theta}$$

By the induction hypothesis, we have an M-proof of a from δ, and an M-proof of θ from b and γ. We must give an M-proof of θ from $i(a,b)$, δ, and γ.

Applying modus ponens to $i(a,b)$ (which is $(A \to B)'$) and the given proof of a from δ, we derive b. Copying the steps of the proof of θ from assumptions b, γ (but changing the justification of the step(s) b from "assumption" to the line number where b has been derived), we derive θ from assumptions $(A \to B)'$,δ, γ, completing the proof of case 1. No double negations are introduced by this step.

Case 2, the last inference in the G-proof is by rule $\Rightarrow\!\to$:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

By the induction hypothesis, we have an M-proof from H1–H4 of b from γ and a. Applying the deduction theorem for H1–H4 with M-proofs, we have an M-proof in H of $i(a, b)$ from γ. But $(A \to B)' = i(a, b)$, completing this case. Note that double negations are not introduced by the deduction theorem if they are not already present, by part (v) of the deduction theorem. (One sees why we must use M-proofs instead of condensed detachment.)

Case 3, the last inference in the G-proof introduces negation on the right:

$$\frac{A, \Gamma \Rightarrow []}{\Gamma \Rightarrow \neg A}$$

By the induction hypothesis, there is an M-proof of $n(a)$ from a and γ. By the deduction theorem for H1–H4 with M-proofs, there is a proof of $i(a, n(a))$ from γ. Hence, it suffices to show that $n(a)$ is derivable from $i(a, n(a))$. This follows from a substitution instance of H3, which is $i(i(x, n(x)), n(x))$, substituting a for x.

Case 4, the last inference in the G-proof introduces negation on the left:

$$\frac{\Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow []}$$

By the induction hypothesis, we have an M-proof of a from γ. We must show that from $n(a)$ and γ, we can deduce b in L, where b is any formula of H. We have $i(a, i(n(a), b))$ as a substitution instance of axiom H4. Applying modus ponens twice, we have the desired M-proof of b from γ, completing case 4.

Case 5, the last inference is by contraction in the antecedent:

$$\frac{C, C, \Gamma \Rightarrow \Theta}{C, \Gamma \Rightarrow \Theta}$$

By the induction hypothesis we have an M-proof of θ from assumptions c, c and γ, which also qualifies as a proof from assumptions c and γ, so there is nothing more to prove.

Case 6, the last inference is by thinning in the antecedent:

$$\frac{\Gamma \Rightarrow \Theta}{C, \Gamma \Rightarrow \Theta}$$

By the induction hypothesis, we have an M-proof from H1–H4 of θ from assumptions Γ'. That counts as an M-proof from assumptions C, γ as well. That completes case 6.

Case 7, the last inference is by interchange in the antecedent. This just means the order of formulas in the assumption list has changed, so there is nothing to prove.

That completes the proof of part (i) of the lemma. Regarding parts (ii) and (iii), by the preceding lemma, any double negations occurring anywhere in the G-proof must occur in the final sequent. No new double negations are introduced in the translation to H, and all the theorems of H that we used have been given double-negation-free condensed-detachment proofs from H1–H4. By Theorem 1, they have double-negation-free M-proofs, too. Although we may not have pointed it out in each case, the argument given produces an M-proof in which any double negations arise from the translations into H of doubly negated subformulas of the final sequent. In particular, if the final sequent contains no double negations, then the M-proof produced also contains no double negations.

Corollary 2 *H is a basis for the implication-negation fragment of intuitionistic logic. That is, every intuitionistically valid formula in this fragment is provable in H.*

Remark. In [4], this lemma was proved for a different axiomatization of the implication-negation fragment of intuitionistic calculus, so this corollary could also be proved by demonstrating the equivalence of the two fragments directly.

Proof. Suppose A is an intuitionistically valid formula containing no connectives other than implication and negation. By Gentzen's cut-elimination theorem, there is a cut-free proof of the sequent $[] \Rightarrow A$ (with empty antecedent). By Lemma 12, A has an M-proof, which in particular is a proof in H.

Remark. The main idea of the corollary is that by the subformula property of cut-free proofs, the cut-free proof contains no connectives other than implication and negation.

Theorem 8 *Suppose H proves b from assumptions δ and neither δ nor b contains double negation. Then there is a condensed-detachment proof of b from H1–H4 and assumptions δ that does not contain double negation.*

More generally, if δ and b are allowed to contain double negation, then there is a condensed-detachment proof of b from H1–H4 and assumptions δ that contains no new double negations. That is, all doubly negated formulas occurring in the proof are subformulas of δ or of b.

Proof. Let $b^0 = B$ be the translation of b into G defined earlier. Double negations in B arise only from double negations in b. Suppose b is provable in H from assumptions δ. By Theorem 1, there is an M-proof of b from δ. By Lemma 11, the sequent $\Delta \Rightarrow B$ is provable in G. Hence, by Gentzen's cut-elimination theorem, there is a proof in G of $\Delta \Rightarrow b$. By the preceding lemma, there is an M-proof of B' from assumptions $\Delta^{0'}$ that contains no new double negations. But by Lemma 8, $B' = b$ and $\Delta' = \delta$. Thus we have an M-proof of b from δ. By the D-completeness of H, Theorem 7, there is also a condensed-detachment proof of b from δ. The second part of Theorem 7 says that we can find a double-negation-free condensed-detachment proof of b from δ. It is important that we are working with M-proofs here because the second part of the D-completeness theorem, about double negations, applies only to M-proofs. That completes the proof.

Theorem 9 *Suppose A is provable from H1–H4 by using condensed detachment as the only rule of inference. Then A has a proof from H1–H4 by using condensed detachment in which no doubly negated formulas occur except those that already occur as subformulas of A.*

Proof. Suppose A is provable from H1–H4 using condensed detachment. Each condensed-detachment step can be converted to three steps by using substitution and modus ponens, so A is provable in H. By the preceding theorem, A has a condensed-detachment proof from H1–H4 in which no doubly negated formulas occur except those that already occur in A. That completes the proof.
Remark. Since the translation back from Gentzen calculus produces M-proofs, we do not need to appeal to D-completeness for arbitrary H-proofs. This is fortunate because we do not know a proof of D-completeness that avoids the possible introduction of double negations, except when restricted to M-proofs.

Corollary 3 *Let T be any set of axioms for intuitionistic propositional logic. Suppose that there exist condensed-detachment proofs of H1–H4 from T in which no double negations occur (except those that occur in T, if any). Then the preceding theorem is true with T in place of H1–H4.*

Proof. Let b be provable from T. Then b is provable from H1–H4 because T is a set of axioms for intuitionistic logic. By the theorem, there is a proof of b from H1–H4 that contains no double negations (except those occurring in b, if any). Supplying the given proofs of H1–H4 from T, we construct a proof of b from T that contains no double negations except those occurring in T or in b (if any). That completes the proof.

11 Double-Negation Postponement

In preceding sections we considered attempts to derive a double-negation-free theorem without introducing double negations in the deduced steps. In this section we consider the question of deducing a theorem that does contain a double negation. In that case, of course, there cannot be a double-negation-free proof, but we show that we can do the next best thing: we can find a proof that is double-negation free until the *very last step*—all the double-negations are introduced at the last application of condensed detachment. This may seem surprising at first, but the proof shows that for many logics, it really is just a simple corollary of our results on double-negation elimination. The result applies to classical logic as axiomatized by L1–L3, for example. If a propositional logic T has this property, we say T satisfies *double-negation postponement*. Note that double-negation postponement is a stronger property than double-negation elimination: double-negation postponement says there is a proof with no double negations except possibly in the axioms or the final conclusion, and if the final conclusion has no double negation, this is just the statement of double-negation elimination. Note also that double-negation postponement implies triple-negation postponement, and so forth, since a triple negation is also a double negation.

We say that T *defines equivalence* if there is a propositional formula $E(x, y)$ such that

$$i(\phi(y), i(E(x, y), \phi(x)))$$

can be proved in T for each formula ϕ. We say that T *defines vector equivalence* if for each positive integer n there is a formula $E_n(x, y)$, with free variables $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$, such that

$$i(\phi(y), i(E_n(x, y), \phi(x)))$$

can be proved in T for each formula ϕ. Clearly, the same formulas satisfy the conditions for $E(x, y)$ and $E_1(x, y)$, and we take these two forms to be synonymous. When x and y are lists (vectors) of several variables, by $n(y)$

we understand the vector $n(y_1), \ldots, n(y_n)$. We may drop the subscript on E_n for notational convenience.

Lemma 13 *Propositional logic L1–L3 can define both equivalence and vector equivalence in such a way that the defining formula $E(x, n(y))$ is double-negation free.*

Proof. Define $x \wedge y$ to be $n(i(x, n(y)))$. Define $E(x, y)$ to be $i(x, y) \wedge i(y, x)$, which is $n(i(i(x, y), n(i(y, x))))$. Then $E(x, n(y))$ is

$$n(i(i(x, n(y)), n(i(n(y), x)))),$$

which is double-negation free. We now proceed by recursion on n to define the formula $E_n(x, y)$ when x and y are vectors of length n. The natural thing is to define $E_{n+1}((x, x_{n+1}), (y, y_{n+1}))$ to be

$$E_n(x, y) \wedge E(x_{n+1}, y_{n+1}),$$

where $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$. Using the definition of conjunction ($\wedge$), we obtain

$$n(i(E_n(x, y), n(E(x_{n+1}, y_{n+1})))),$$

and using the definition of equivalence on the right, we get

$$n(i(E_n(x, y), n(n(i(i(x_{n+1}, y_{n+1}), n(i(y_{n+1}, x_{n+1}))))))).$$

A double negation has been introduced, which is undesirable. We therefore cancel that double negation and define instead:

$$\begin{aligned} &E_{n+1}((x, x_{n+1}), (y, y_{n+1})) \\ &\quad := n(i(E_n(x, y), i(i(x_{n+1}, y_{n+1}), n(i(y_{n+1}, x_{n+1}))))). \end{aligned}$$

Now we can prove by induction on n that $E_n(x, y)$ is double-negation free for each n. We have already done the base case, and the definition makes the induction step apparent.

It remains to verify that L1–L3 proves $i(\phi(y), i(E(x, y), \phi(x)))$ for each formula ϕ and each n, where $x = x_1, \ldots, x_n$. Since L1–L3 is an axiomatization of classical logic, we can appeal to the completeness theorem: it suffices to show that $i(\phi(y), i(E(x, y), \phi(x)))$ is valid under each assignment of truth values to the variables x and y. For this it suffices to show that if $E(x, y)$ is satisfied under a given truth assignment, then x_i gets the same truth value as y_i for each $i = 1, \ldots, n$. This we prove by induction on n. For the base case we have $E(x, y) = n(i(i(x, y), n(i(y, x))))$, and a four-line truth table can be used to verify the claim; alternatively one can verify the truth table of $x \wedge y$ as defined above and the truth-table equivalence of $E(x, y)$ and $i(x, y) \wedge i(y, x)$. The induction step is apparent from the definition of E_{n+1} in terms of E and E_n. That completes the proof of the lemma.

Lemma 14 *L1–L3 proves every instance of $E(x, x)$, where $x = x_1, \ldots, x_n$.*

Proof. It probably is possible to show how to actually construct such a proof, proceeding by induction on n, but it is not necessary because L1–L3 is an axiomatization of classical logic, and all these statements are valid, as is easily proved by induction on n. By the completeness theorem and D-completeness, valid statements are provable in L1–L3. That completes the proof.

Theorem 10 (Postponement of double negation) *Let T be any propositional logic such that (1) T satisfies double-negation elimination, (2) T can define equivalence and vector equivalence in such a way that $E(x, n(y))$ is double-negation free, and (3) T can prove every instance of $E_n(x, x)$. Then T satisfies double-negation postponement. In particular, L1–L3 satisfies double-negation postponement.*

Proof. We verified earlier that L1–L3 satisfies hypotheses (1), and in the lemmas we verified hypotheses (2) and (3). Hence, it suffices to derive double-negation postponement from those hypotheses.

For simplicity, we first consider the case of a theorem A of T that contains a single double negation, but no triple negation, and no other double negation (though it may contain multiple copies of that same doubly negated formula). Let the doubly negated formula be $n(n(P))$, a subformula of the theorem A. Let $\phi(x)$ be the formula resulting from A by replacing $n(P)$ by a new variable x, not occurring in A. Thus A is $\phi[x := n(P)]$, or for short $\phi(n(P))$, and ϕ is double-negation free. We must show that A has a proof in which double negation enters only at the last step. Since T can define equivalence, there is a proof of

$$i(\phi(n(P)), i(E(x, n(P)), \phi(x))).$$

By hypothesis, there is some proof of A (which is $\phi(n(P))$) in T. Hence there is a proof of

$$i(E(x, n(P)), \phi(x)).$$

Since $E(x, n(y))$ is double-negation free and P and ϕ are double-negation free, the displayed formula is double-negation free. Since T satisfies double-negation elimination, there is a double-negation-free proof of that formula. By hypothesis (3), there also is a proof in T of $E(n(P), n(P))$. Since we assumed that A has no triple negation, the formula P is not a negation, and since we assumed that $n(n(P))$ is the only double negation in A, P is double-negation free. Applying double-negation elimination again, we have a double-negation-free proof of $E(n(P), n(P))$. Applying condensed detachment to this formula and $i(E(x, n(P)), \phi(x))$, we obtain the conclusion $\phi(n(P))$ as desired. That completes the proof in case A has no triple negations and only one double negation. Note that only the definability of equivalence, not vector equivalence, was used so far.

Next we consider how to relax those assumptions by introducing more than one extra variable x. We do so recursively. Namely, we find a formula P that is double-negation free, is not itself a negation, and occurs doubly negated in A. We introduce a new variable x in place of $n(P)$ as above, so that A is $\psi[x := n(P)]$ for some ψ. If there is any double negation in A, then such a formula P can be found. After replacing $n(P)$ by x to obtain ψ, we proceed recursively to make similar replacements in ψ. Since each new variable reduces the number of doubly negated subformulas, this process will terminate after introducing variables $x_1, \ldots, x_n$ for some n, and A will be of the form $\phi[x_1 := n(P_1), \ldots x_n := n(P_n)]$. If we write $n(P)$ for $n(P_1), \ldots, n(P_n)$ and x for the vector $x_1, \ldots, x_n$, we can use the more compact notation $\phi[x := n(P)]$. We can now repeat the proof in the preceding paragraph verbatim, but now it applies to the general case. We need the definability of vector equivalence to show that $E(x, n(y))$ is double-negation free when x and y are vectors and to show that $E(n(P), n(P))$ is double-negation free. That completes the proof.

Example. Frege's propositional logic includes the axioms $i(n(n(x)), x)$ and $i(x, n(n(x)))$. By our theorem, there must exist proofs of each of these axioms from L1–L3 that involve double negation only in the conclusion. Indeed, it is straightforward to find such proofs with Otter. We use the second example, $i(x, n(n(x)))$, to illustrate the method implicit in the proof of the double-negation postponement theorem. That method tells us to introduce a new variable y and seek double-negation-free proofs of $i(E(y, n(x)), i(x, n(y)))$ and $E(n(x), n(x))$. If such proofs are found, then the desired double-negation-postponed proof of $i(x, n(n(x)))$ requires only one more step. Writing out the definition of E, the two formulas to be proved are

$$i(n(i(i(y, n(x)), n(i(n(x), y)))), i(x, n(y)))$$

and

$$n(i(i(n(x), n(x)), n(i(n(x), n(x))))).$$

In principle, we could use the proof of the double-negation elimination theorem to find double-negation free proofs of these theorems and then construct the double-negation-postponed proof of $i(x, n(n(x)))$. This is not necessary, however, since we found the desired proof in this case quite easily without relying on the method. After all, the theorems proved in this paper were designed not so much to supplement our theorem-proving strategies as to justify them.

Remarks. Besides L1–L3, the other examples of systems discussed in this paper were infinite-valued logic A1–A4 and the intuitionistic system H, both of which are formulated with only implication and negation as connectives. Whether double-negation postponement is true for A1–A4 or for H itself, we do not know. If H admits double-negation postponement, it will require a different proof. It is known (see [11], p. 253) that H cannot define conjunction, so the definability of equivalence cannot be carried out for H in the same way as for L1–L3. Indeed, Albert Visser has proved (private correspondence) that H cannot define equivalence (and his proof reproves the result for conjunction too).

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