Constructivity, Computability, and the Continuum

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Two fundamental principles

- (Church's thesis) *Every real number can be computed to any desired approximation by an algorithm.*
- (Geometric completeness) *The points on a line segment correspond to real numbers in an interval.*

Is it really possible that all those gaps are filled up with computable numbers?

Alonzo Church

The constructivist reply

- You can’t point out an unfilled gap, since the computable numbers are not recursively enumerable.

Fine, but you can’t point out the dirt after it’s swept under the carpet, either.

The Continuum in the History of Logic

- Euclid and Zeno
- Staudt (*Geometrie der Lage,* 1847)
- Dedekind and Cantor
- Klein (1873)
- Pasch (1882)
- Veronese, Enrique, Pieri, Padoa
- Peano
- Hilbert's *Grundlagen der Geometrie* 1899

Hilbert's viewpoint

- Wir denken uns drie verschiedene Systeme von Dingen…
- *With this, the umbilical cord between reality and geometry is severed [Freudenthal]*

Brouwer

- Wrestled with the *actual* continuum
- Invented a theory of choice sequences
  0.33443334444433… digits chosen by free will
- This led to the continuity principle: all functions are continuous, since they need to be computable on choice-sequence arguments.
- Flatly contradicted classical mathematics.

Hermann Weyl

- At the center of my reflections stands the conceptual problem posed by the continuum—a problem which ought to bear the name of *Pythagoras* and which we currently attempt to solve by means of the arithmetical theory of irrational numbers.

  -- preface to *Das Continuum*, 1917

Recursion theory and the continuum

- Turing's original paper mentioned *computable numbers*.
- A recursive real is given by a recursive sequence of rationals converging at a specified rate, e.g.

$$x_j - x_k \leq \frac{1}{2^j} + \frac{1}{2^k}$$

Kleene's singular tree

- *There is an infinite binary recursive tree with no infinite recursive path.*
- *König's lemma is false in the recursive reals.*
- *K = { t of length n : for each k < n, n steps of computation do not reveal that*

$$(t)_k \neq \{k\}(k)$$

- *Any path would separate the two recursively inseparable sets*

  {n: {n}(n) = 0} and {n : {n}(n) is not 0}

Lacombe's Singular Cover

- The set of recursive members of 2^N has (classically) measure zero.
- Let *k* be given; we cover 2^N with neighborhoods A_n of total measure less than $1/2^k$. Enumerate indices y_n of partial recursive functions y_n whose first $k+y$ values are defined. Let be the set of functions agreeing with y_n for the first y_n+k values.
- Measure of A_n is $1/2^j$ where $j = y_n+k$.

Specker Sequences

- The final public-relations disaster for recursive analysis
- Bounded recursive monotone sequence of recursive reals with a non-recursive limit
- Define x_n so that the k-th digit is 3 if n steps of computation of $\{k\}(k)$ do not yield a value, or 4 if they do.
- The limit number solves the halting problem.

Recursion theory and the Continuum

- Recursive binary trees have Delta_0^2 paths
- Hyperarithmetic binary trees have hyperarithmetic paths
- Connection between fullness of continuum and our ability to define reals by quantification over natural numbers.

Options at the time of Brouwer's death in 1966

- Accept Brouwer's intuitionism, give up most of mathematics, and give up talking to most mathematicians.
- Accept Church's thesis, give up analysis, and give up talking to all mathematicians except a few Russian constructivists.
- Reject constructive mathematics entirely.

Bishop's constructive mathematics (1967)

- No non-classical principles, so consistent with classical mathematics.
- Nevertheless, also consistent with Church's thesis.
- Existence theorems proved by giving constructions.
- A substantial body of mathematics was developed.

Bishop's measure theory

- [0,1] has measure 1
- If X has positive measure, we can construct an element of X (basic theorem of measure theory).
- Yet Bishop's work is consistent with CT
- What about the singular cover?
- The explanation of this is in my paper but can't be covered in a 25-minute talk.

Geometric completeness

- Definition: *X has measure at most t if X is covered by a union of (a sequence of) neighborhoods such that the sum of any finite number of those neighborhoods is less than or equal to t.*
- *Fullness Principle (FP):*

  If [0,1] (or 2^N) has measure at most t, then t is greater than or equal to 1.

Measure of the recursive reals

- Recursive reals have measure at most epsilon, for each positive epsilon.
- This isn’t sufficient to show they have measure less than 1, in Bishop’s measure theory.
- The intervals A_n in Lacombe’s cover overlap, and their total measure is not computable.   At any point in computing the limit, the value so far can take an unpredictable jump.

FP is justified by the geometric completeness principle

- FP says that there are enough points to fill up a geometric line segment.
- Yet it avoids asserting the existence of any particular real numbers.
- Via Lacombe's singular cover, it refutes CT
- It also refutes: *there exists a real number x such that every real number is recursive in x*.

Intuitionistic Weak König's Lemma IWKL

- Every infinite binary tree is not well-founded.
- Every well-founded binary tree is not infinite
- There are no well-founded infinite binary trees
- Trees are given by the complements of the union of a sequence of neighborhoods.

Relations of IWKL and FP

- IWKL implies FP
- Whether FP implies IWKL is open.
- One attempt to prove FP implies IWKL fails, because the cover associated with Kleene's singular tree has measure 1.   (That cover consists of neighborhoods given by sequences *t* not in the tree K, but all their initial segments are in K.)
- So Kleene's and Lacombe's constructions are essentially different.

Comparison to Brouwer's fan theorem

- We have considered trees whose complements are a union of neighborhoods.
- The corresponding restricted version of the fan theorem is Heine-Borel's theorem for $2^{\mathbb{N}}$:
- (HB) *Every well-founded binary tree is finite.*
- The converse of HB is IWKL.
- An additional intuition beyond geometric completeness would be needed to justify HB.

Metamathematical Analysis of FP

- Relative to HA-omega, arithmetic of finite types.
- Disjunction and existence properties.
- Numerical existence property.
- Provably total functions are recursive (Church’s rule)
- Conservative over HA

Method of Proof for the numerical existence property and Church's rule

- Modified q-realizability. FP proves its own q-realizability.
- Standard technique from Troelstra (SLN 344)

Proof of conservativity over HA

- Formulate a stronger principle NPE
- NPE equivalent to FP using AC-1-0.
- Interpret HA-omega using Kleene's countable functionals; A goes to A*, say.
- With a suitable notion of forcing, NPE* can be made generically valid.
- Hence NPE is conservative over HA.

What is NPE?

- A path-ender for a tree given by a sequence of neighborhoods A_n is a function e such that for every function g in 2N, $e(g)$ is a pair (n,k) such that the initial segment of g of length k belongs to A_n.
- NPE (No path-enders) says that there exists a functional F such that for every functional e of the type of a path-ender, $F(e)$ is a function g such that $e(g)$ is not a pair (n,k) such that the initial segment of g of length k belongs to A_n.
- That is, $F(e)$ shows that e is not a path-ender for the tree given by the sequence A_n.

Proof of conservativity continued

- AC* is "continuous choice" CC
- HA-omega + CC is conservative over HA by the realizability-forcing technique in Chapter XV of Beeson [1985]
- If FP proves an arithmetic B, then HA-omega + CC proves B* is generically valid; hence HA-omega proves B*. But B* is B, so HA-omega proves B.
- Hence HA proves B.

What is mathematics about?

Freudenthal: *Whether one believed with Kant that axioms arose out of pure contemplation, or with Helmholtz that they were idealizations of experience, or with Riemann that they were hypothetical judgements about reality, in any event nobody doubted that axioms expressed truths about the properties of actual space and were to be used for the investigation of properties of actual space.*

Failure of geometry in the large

- Development of non-Euclidean and Riemannian geometry and their application to the general theory of relativity dealt a death blow to this idea.

Failure of geometry in the small

- Destruction of Kantian ideas by physics should be more widely known.
- Assume that space is coordinatizable.
- Assume particles can’t move faster than light.
- Assume the Schwarzschild radius equation

  $r = MG/c^2$ for black hole creation
- Assume the uncertainty principle.
- Assume spontaneous particle-antiparticle virtual pair creation.
- A contradiction follows in one Powerpoint slide.
- Credit: calculation shown to me by Bob Piccioni.

The Planck Length

- The idea of the contradiction is this:
- A virtual particle of mass M is created, travels a distance r and back before it is annihilated. It has to take time at least 2r/c to do that; its energy is Mc^2 so the uncertainty principle gives $2Mrc > \hbar$.
- To prevent the mass from collapsing into a black hole, according to the Schwarzschild equation M cannot be too large: $M < rc^2/2G$.
- Put that into the first equation, get

$$r > \mathrm{sqrt}(\hbar\, G/\, c^3) = 1.616 \times 10^{-33}\ \ \mathrm{cm}$$

So much for Kant

- Helmholtz vindicated. Our axioms are *idealizations of experience.*
- We can "zoom in" on a line segment, straighten out any imperfections, and repeat.
- In physical reality we can do this only about 100 times before we reach the Planck length.
- In our mind's eye we can do it as many times as there are positive integers.

Conclusions

- Church’s thesis and the Geometric Completeness Principle are indeed contradictory (since FP refutes CT)
- FP has the metamathematical properties one expects of a constructive theory.
- The geometric continuum is filled with non-recursive members, even though we cannot prove their individual existence.
- Our intuition of their existence is based on idealizations of experience, not on physical reality.