The Parallel Postulate in Constructive Geometry

Independence Results

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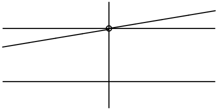
Postulates *vs* Axioms (according to Geminus and Dehn)

- Postulates set forth our abilities to make certain constructions.
- Axioms merely state (static) properties
- Aristotle and Proclus offer different explanations of the difference, but I like this explanation.
- The idea is not Dehn's but is already attributed to Geminus by Proclus.
- Example: (Postulate 3) To describe a circle with any center and distance.

Euclid's Parallel Postulate

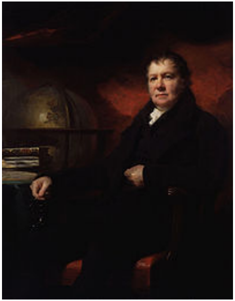
Euclid's Postulate 5 [Heath translation]: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

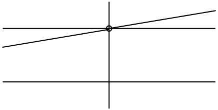


The two lines must meet.

Playfair's Parallel Axiom

Definition: K is parallel to L if K does not meet L, i.e. no point lies on both K and L. (John Playfair, 1795; Proclus, 450) Given a line L and a point P not on L, there exists exactly one parallel to L through P.





The two lines can't fail to meet.

Euclid 5 implies Playfair

Constructively, Euclid 5 implies Playfair. Given line K through P not on L, let M be the perpendicular to L through P. If K does not meet L, then by (the converse of) Euclid 5, the interior angles on each side of M are not less than two right angles. It follows that M is perpendicular to K as well as L. Hence there is only one possibility for K, QED.

But there is no obvious way to derive Euclid 5 from Playfair constructively.

A third version of the parallel postulate

Strong parallel postulate: Let K be a line through a point P not on line L. If K is not parallel to L then K meets L.

This postulate implies Euclid 5, but the converse is not obvious. Constructively, we might know that K is not parallel to L without knowing on which side the interior angles are less than two right angles.

The main points of today's talk

We obtain the following results:

- Playfair does not imply Euclid 5 in constructive geometry.
- Euclid 5 does not imply the strong parallel postulate in constructive geometry.

Constructive geometry

Does *constructive* refer to the use of intuitionistic logic?

Or does it refer to geometrical constructions with ruler and compass?

What is the relation between these two?

In our constructive geometry, they are closely related: things proved to exist can be constructed with ruler and compass.

Primitive Constructions

- *Line* (A,B)
- *Circle* (A,B) (center A, passes through B)

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- *IntersectLineCircle2* (A,B,C,D)
- *IntersectCircles1* (c_1,c_2)
- *IntersectCircles2* (c_1,c_2)

How the sorts mix

You can also write

$$\mathit{IntersectLines}\,(L, K)$$

if L and K are lines. Then

$$\mathit{IntersectLines}\,(A, B, C, D) = \mathit{IntersectLines}\,(\mathit{Line}\,(A, B), \mathit{Line}\,(C, D))$$

and so on.

First order theories of geometry

- Angles can be treated as ordered triples of points.
- Rays and segments are needed only for visual effect; for theory we need only points, lines, and circles.
- We don't even need lines and circles; every theorem comes down to constructing some points from given points, so that the constructed points bear certain relations to the original points.
- The relations in question can be expressed in terms of *betweenness* and *equidistance*.
- We can also use a three-sorted language, for points, lines, and circles.
- These theories are all mutually interpretable and those with larger languages are conservative extensions.

Tarski geometry

Just to avoid confusion: today we are concerned with "elementary" geometry in the sense that only line-circle and circle-circle continuity are used. Hilbert's geometry included a second-order continuity axiom, essentially requiring that Dedekind cuts be filled. "Tarski geometry" is a first-order theory with a continuity schema, essentially requiring that first-order definable Dedekind cuts be filled. Sometimes "elementary" means first-order, and Tarski wrote a famous paper, *What is Elementary Geometry*, in which "elementary geometry" meant Tarski geometry. But "elementary" can also refer to the *Elements* of Euclid, which is a weaker theory.

Models of the Elementary Constructions

- The "standard plane" $\mathbb{R}^2$
- The "recursive plane". Points are given by recursive functions giving rational approximations to within $1/n$.
- The minimal model, the points constructible by ruler and compass
- The algebraic plane, points with algebraic coordinates
- The Poincaré model. These constructions work in non-Euclidean geometry too.

Book I, Prop. 2

- Euclid's compass is "collapsible"
- You cannot use it directly to construct Circle3(A,B,C), the circle with center A and radius BC.
- Book I, Prop. 2 is intended to show that Circle3 need not be assumed, because
- Given A, B, and C, we can construct a point D with $AD = BC$.

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- Given A, B, and C, we can construct a point D with $AD = BC$.
- But Euclid's construction assumes not only $B \neq C$, but also $B \neq A$, and the point constructed does not depend continuously on B as B tends to A.

Euclid's proof of I.2 requires a case split

- This is a bad omen for constructive geometry, because computable points must depend continuously on their parameters.
- Euclid doesn't argue by cases. He just omits the "trivial" case.
- Euclid was criticized already in 450 AD by Proclus for not taking sufficient care with argument by cases.
- Proclus considered eight "cases" (different diagrams) including the case A=B, which Heath thinks is superfluous.

Uniform I.2

- The uniform version of this proposition says that, given A, B, and C, with $B \neq C$, we can construct $D = e(A, B, C)$ with $AD = BC$, without assuming $A \neq B$.
- Given such a term e, we could define *Circle3* $(A, B, C) =$ *Circle* $(A, e(A, B, C))$.
- Given *Circle3* , we could define $e(A, B, C) =$ *pointOnCircle*(*Circle3* (A, B, C).
- Having *Circle3* is equivalent to "realizing" uniform I.2.
- In constructive geometry, we take *Circle3* as primitive, so Euclid I.2 is essentially taken as an axiom.

Intuitionistic Geometry

- Use intuitionistic logic
- In intuitionistic logic, we do not have $a < b \vee a = b \vee b < a$ for points on a line.
- In view of that, several of Hilbert's axioms are not correct with intuitionistic logic.
- Reductions to field theory need reconsideration.

Right and left turns

We have a problem in distinguishing the two intersection points *IntersectCircles1* (C_1, C_2) and *IntersectCircles2* (C_1, C_2). Suppose C_1 has center A and C_2 has center B, and P is an intersection point. Then, we consider whether the triple ABP is a "left turn" or a "right turn". That is how we shall distinguish the two intersection points.

In order to be able to do that, we have to be able to define what it means for three points to form a "left turn" or a "right turn". As far as I know, this problem has not been solved in the geometrical literature. It was not very difficult, but it was also not quite trivial.

Defining right and left turns

One has three fixed non-collinear points α, β, and γ. One arbitrarily defines $\alpha\beta\gamma$ to be a right turn and $\alpha\gamma\beta$ to be a left turn. One then needs to be able to define what it means for ABC to be a right or left turn in general and prove some basic lemmas about these concepts. The idea is that there are certain operations or "moves" one can perform on angles (triples of points) without changing the handedness; and a chain of no more than 20 such moves can transform any angle into any other angle of the same handedness. The details can be found in my first paper on constructive geometry.

Right turn and left turn

One could add predicates $R(A, B, C)$ and $L(A, B, C)$, but one can instead just regard $R(A, B, C)$ as an abbreviation for

$$\mathit{IntersectCircles1}\,(\mathit{Circle}\ (A, C), \mathit{Circle}\ (B, C)) = C$$

since one would otherwise have to add an axiom stating that these two are equivalent.

Then one needs axioms about R corresponding to the "moves" that preserve handedness. For example, if $R(A, B, C)$, and D is between A and B, then $R(D, B, C)$. These are technically part of the axiomatization of *IntersectCircles1* and *IntersectCircles2*. That their interpretations in a more conventional geometry are provable shows that handedness is indeed first-order definable.

Plan for a constructive geometry

- Take the primitive constructors discussed above, including *Circle3*
- Consider the axioms from a Hilbert-style axiomatization, such as the one in Greenberg's textbook.
- Using the function symbols for the constructors, express the axioms in a quantifier-free and disjunction-free way.
- For continuity, take only line-circle continuity and circle-circle continuity.
- See if enough geometry can be done with these axioms.

What is "enough geometry"?

- All of Euclid
- Definability of addition and multiplication on signed segments without using non-constructive case splits

Does Euclid use the law of the excluded middle?

Euclid's proofs have been analyzed in detail by Avigad, Dean, and Mumma, and they conclude:

> *Euclidean proofs do little more than introduce objects satisfying lists of atomic (or negation atomic) assertions, and then draw further atomic (or negation atomic) conclusions from these, in a simple linear fashion. There are two minor departures from this pattern.*

An example where Euclid deviates from intuitionistic logic

Prop. I.6, whose proof begins

> *Let ABC be a triangle having the angle ABC equal to the angle ACB. I say that the side AB is also equal to the side AC. For, if AB is unequal to AC, one of them is greater. Let AB be greater, ...*

Since the *conclusion* doesn't have a disjunction, we can just push double negation through this case split, and then use $\neg x \neq y \supset x = y$.

Euclid is almost completely constructive

Avigad, Dean, and Mumma summarize Euclid's few non-constructive arguments this way:

> *Sometimes a Euclidean proof involves a case split; for example, if ab and cd are unequal segments, then one is longer than the other, and one can argue that a desired conclusion follows in either case. The other exception is that Euclid sometimes uses a* reductio*; for example, if the supposition that ab and cd are unequal yields a contradiction then one can conclude that ab and cd are equal.*

Euclid can be made completely constructive

Using the stability of equality $\neg\neg x = y \supset x = y$ and Markov's principle $\neg\neg x < y \supset x < y$ (expressed using betweenness), we can simply push double negation through Euclid's few arguments by cases, except for the extended (uniform) version of Prop. I.2.

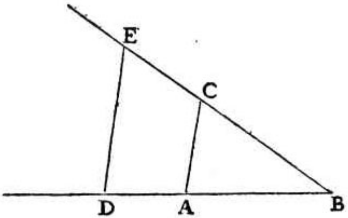
I don't know whether there is a proposition in Euclid that actually needs Markov's principle. That would be an argument by cases whose conclusion is an inequality (asserting involving betweenness) rather than an equality.

Descartes

La Geometrie (1637) introduced the idea of performing arithmetic on (the lengths of) segments by geometrical construction.

Page Two of *La Geometrie*

La Multi-plication.

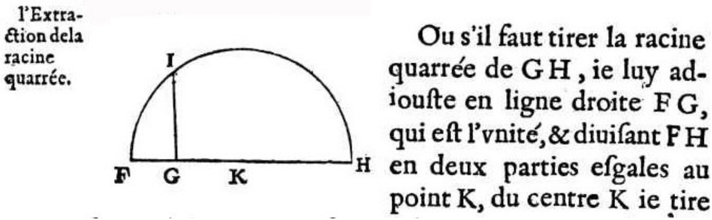


Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

La Diuiſion.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

Page Two of *La Geometrie*



l'Extra-
ction dela
racine
quarrée.

Ou s'il faut tirer la racine quarrée de G H, ie luy adiouste en ligne droite F G, qui est l'vnité, & diuisant F H en deux parties esgales au point K, du centre K ie tire

In this picture, FG is unity and triangle IFG is similar to HIG since angle FIH is a right angle. Hence $IG/1 = GH/IG$ so $IG^2 = GH$.

Continuity in defining arithmetic

Descartes's methods work only for arithmetic on segments representing positive numbers. It is a bit tricky to define addition and multiplication to work for *signed* numbers represented by a point on a line with an arbitrarily chosen "zero" point.
This was the major technicality in my first paper on constructive geometry. Curiously, the theory of right and left turns was also crucial to solving these problems.

Intuitionistic Euclidean Geometry of Constructions IECG

The result of this approach to axiomatizing geometry is a quantifier-free, disjunction-free system, with function symbols *IntersectCircles1*, etc., and intuitionistic logic, such that

- All of Euclid is formalizable in ECG in a natural way.
- Arithmetic on signed segments can be defined in ECG using only constructive logic
- With classical logic, ECG is equivalent to the usual theories of geometry (with line-circle and circle-circle continuity)

Features of IECG

- Euclid I.2 taken as an axiom (Circle from point and radius)
- Strong parallel axiom
- Quantifier-free, disjunction-free
- Markov's principle for betweeness (amounts to $\neg x \leq 0 \supset x > 0$)

Constructive logic and Euclid's constructions

The main result of my first paper on constructive geometry is

What is constructively proved to exist in EGC is constructible with straightedge and compass.

This is an easy consequence of Gentzen's cut-elimination theorem, because of the care taken to formulate EGC in a quantifier-free, disjunction-free axiomatization.

A more precise statement

Theorem. Suppose IEGC proves

$$\forall x\,(P(x) \supset \exists y\, A(x,y))$$

with P negative. Then there exist a term $t(x)$ such that IEGC proves

$$\forall x\,(P(x) \supset A(x,[y := t(x)]))$$

Here x can stand for several variables.

Terms of IEGC correspond directly to (uniform) Euclidean constructions; in particular they are all continuous on their domains.

Constructions and classical logic

Theorem. Suppose EGC with classical logic proves

$$\forall x\, (P(x) \supset \exists y\, A(x, y))$$

with P and A quantifier-free. Then there exist terms $t_1(x), \ldots, t_n(x)$ of EGC such that EGC proves

$$\forall x\, (P(x) \supset A(x, [y := t_1(x)]) \vee \ldots \vee A(x, [y := t_n(x)])$$

Proof. By Herbrand's theorem.

Axioms for euclidean fields

$x \neq 0 \supset \exists y\,(x \cdot y = 1)$	EF1
$P(x) \wedge P(y) \supset P(x+y) \wedge P(x \cdot y)$	EF2
$x + y = 0 \supset \neg(P(x) \wedge P(y))$	EF3
$x + y = 0 \wedge \neg P(x) \wedge \neg P(y) \supset x = 0$	EF4
$x + y = 0 \wedge \neg P(y) \supset \exists\, z(z \cdot z = x)$	EF5
$\neg\neg P(x) \supset P(x)$	EF6, or Markov's principle

Coordinatization

- Every model of IECG is a plane over some euclidean field.
- Because of quantifier elimination (Tarski) every real-closed field gives a model of Tarski geometry.
- Euclidean fields (every positive element has a square root) correspond to the geometry of constructions.

A problem of Tarksi

- Is the geometry of constructions decidable?
- That is, the theory of Euclidean fields (ordered fields in which positive elements have square roots)?
- Ziegler (1980) says not. Indeed any finitely axiomatizable field theory that has $\mathbb{R}$ or the p-adics as a model is undecidable.
- His proof is only 11 (difficult) pages.
- I have translated this paper if anyone wants an English version.

Another problem of Tarski

- Is the smallest Euclidean field $Q(\sqrt{\ })$ decidable?
- Goes beyond J. Robinson's famous results for Q and the algebraic number fields, because $Q(\sqrt{\ })$ is not of finite degree over Q.
- Still an open problem.

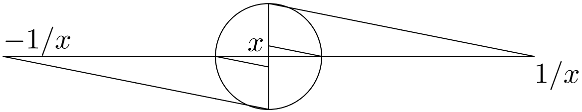
Constructive ordered fields

The axiom about reciprocals in an ordered field can be formulated three ways.

- Nonzero elements have reciprocals
- Positive elements have reciprocals
- Elements without reciprocals are zero

These axioms correspond, respectively, to geometries satisfying the strong parallel axiom, Euclid 5, or Playfair.

Division and parallels



The circle has radius 1. The slanted lines are parallel. $1/x$ is defined if and only if the horizontal line intersects the long slanted line. If we know the sign of x then Euclid 5 suffices; the vertical line is a transversal and on one side the interior angles are less than two right angles.

Strong parallel postulate implies Euclid 5 implies Playfair

- If nonzero elements have reciprocals then positive elements have reciprocals.
- Therefore the strong parallel postulate implies Euclid 5.
- If positive elements have reciprocals then elements without reciprocals are zero. (since if x has no reciprocal then neither x nor $-x$ is positive, so x is zero).
- Therefore Euclid 5 implies Playfair.

Independence results reduced to field theory

Therefore, the independence results we stated reduce to the corresponding results in field theory. (The loose talk about "models" can be replaced by formal interpretations of the geometrical theories into the corresponding field theories, and vice-versa.) We must show that, with the aid of the axioms of ring theory, the implications from positive-reciprocals to non-zero reciprocals, and from no-reciprocals implies zero to positive-reciprocals, and not provable with intuitionistic logic.

Kripke models

Our technique is Kripke models. In the context of ordered ring theory, a Kripke model is given by collection of rings R_α, where the index α comes from some partially ordered set $(D, \preceq)$. Usually D is a tree. The rings R_α have to satisfy the condition that R_α is a sub-ordered-ring of R_β if $\alpha \preceq \beta$.

Satisfaction in Kripke models

Saul Kripke gave a definition of $\alpha \models \phi$, where ϕ can contain constants for elements of R_α. It is like the usual definition in classical logic for $\wedge$, $\vee$, and $\exists$, but

- $\alpha \models A \supset B$ iff and only if whenever $\alpha \preceq \beta$ and $\beta \models A$ then for some γ with $\beta \preceq \gamma$, we have $\gamma \models B$.
- $\alpha \models \neg A$ if and only if for all β with $\alpha \preceq \beta$, we do not have $\beta \models A$.
- $\alpha \models \forall x A$ if and only if, whenever $\alpha \preceq \beta$ and $x \in R_\beta$, we have $\beta \models A(x)$.

How to use Kripke models

Theorems of intuitionistic logic are true in all Kripke models. Hence to show that A does not imply B, it will suffice to give a Kripke model of A that does not satisfy B.

Example of a Kripke model

Let's practice by showing that ordered ring theory does not prove $\forall x(\neg P(x) \vee \neg P(-x))$. Let the partially ordered set D be a tree with root 0, and all branches of length 1, so all the other nodes α are "just above" 0 and they are not comparable to each other. Let's take such a node for some infinite set Ω of irrational real numbers α. Let's assume Ω is dense in $\mathbb{R}$.

Let each of the rings R_α be the ring of rational functions over the reals, $\mathbb{R}(x)$. They will differ only in the interpretation of the positivity predicate $P(x)$. At the root node, we take f to be positive if and only if it is positive definite, i.e. $f(x) > 0$ for all x. Classically, R_0 is not even an ordered ring.

When α is not the root, we take $P(f)$ to mean that $f(\alpha) > 0$. This does turn R_α into a classical ordered ring.

Verification of the ordered ring axioms

We have to check that not both f and $-f$ are positive at the root. That would mean that both f and $-f$ are positive definite, which is absurd.

Next we check (at the root node) that if $\neg P(f)$ and $\neg P(-f)$ then $f = 0$. If $\neg P(f)$ holds at the root node, that means that $f(\alpha) \leq 0$ for each α in Ω. If $\neg P(-f)$ holds at the root node, that means that $-f(\alpha) \leq 0$ for each α. If both hold then $f(\alpha) = 0$ for each α in Ω. But since f is a polynomial, and Ω is infinite, f must be zero.

Our warm-up independence result

Now we check that $\forall x(\neg P(x) \vee \neg P(-x))$ is not satisfied at the root. Let the variable x be interpreted as the polynomial x. By the definition of satisfaction for $\vee$, either the root satisfies $\neg P(x)$, or it satisfies $\neg P(-x)$. Suppose it satisfies $\neg P(x)$. Then for no α do we have $\alpha > 0$. But by our choice of Ω, this is absurd. Similarly, if the root satisfies $\neg P(-x)$, then for no α do we have $-\alpha > 0$; but again, this is absurd. QED.

A certain ring of functions $\mathcal{A}$

Let $\mathbb{K}$ be the field of "constructible numbers", which is the least subfield of $\mathbb{R}$ closed under taking square roots.
Let C_0 be the ring of polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ with coefficients in $\mathbb{K}$.
Define C_{n+1} to be the least ring of real-valued functions containing C_n together with all square roots and reciprocals of positive semidefinite members of C_n. These square roots and reciprocals are defined on $\mathbb{R}$ except at finitely many points, as we will soon see.
Define $\mathcal{A}$ to be the union of the C_n.

Examples of functions in $\mathcal{A}$

For example, the functions $\sqrt{1+t^2}$ and $1/(1+t^2)$ are in $\mathcal{C}_1$, and

$$\sqrt{\sqrt{1+t^2}+\sqrt{1+t^4}}+\frac{1}{1+t^2}$$

is in C_2. The square root of that function is in C_3.

A suitable set Ω

By induction on n, we see that each f in $\mathcal{A}$ has a convergent Pusieux series (a power series in some rational power of $t - a$) at each point a, and also at ∞. Hence (if f is not identically zero) the zeroes of f are isolated, and also isolated from infinity. By compactness, then, f has finitely many zeroes and singularities. Hence there is a countable set of reals that includes all the zeroes and singularities of all the functions in $\mathcal{A}$.
Define Ω to be the complement of that set; thus for each f in $\mathcal{A}$, if $f(x)$ is zero for any x in Ω, then f is identically zero. Note that, since the complement of Ω is countable, Ω is dense in $\mathbb{R}$.

A fancier Kripke model

As in our practice problem, the Kripke model will be a tree, with 0 for the root, and incomparable nodes α for each α in Ω, each of which lies above the root. The ring at the root is $\mathcal{A}$, with $P(x)$ in $\mathcal{A}$ interpreted as "x is positive semidefinite but not identically zero." Then, although $\mathcal{A}$ is not a field, it does have reciprocals of elements satisfying $P(x)$.

At each node α, the ring $\mathcal{A}_\alpha$ is the quotient field of $\mathcal{A}$, whose elements we can take to be of the form x/y with $y(\alpha) > 0$, and interpret $P(x/y)$ as $x(\alpha) > 0$. This makes $\mathcal{A}_\alpha$ isomorphic to the least euclidean subfield of $\mathbb{R}$ containing α. The isomorphism sends x/y to $x(\alpha)/y(\alpha)$. It is a one-to-one because if $x(\alpha) = 0$ then x is (identically) zero, by our choice of Ω.

Monotonicity of $P(x)$

If $P(x)$ holds in $\mathcal{A}$, then x is positive definite, so $x(\alpha) > 0$, so $P(x)$ holds in $\mathcal{A}_\alpha$. Of course, $P(x)$ also holds in $\mathcal{A}_\alpha$ for many functions x that are not positive definite, but that is all right. The ring axioms are satisfied in this Kripke structure, since all the $\mathcal{A}_\alpha$ and $\mathcal{A}$ are rings. Since the $\mathcal{A}_\alpha$ are fields, the field axioms hold automatically there; we only need to consider whether they hold at the root node $\mathcal{A}$.

Axiom EF2

Consider Axiom EF2, which says that sums and products of positive elements are positive. This holds at $\mathcal{A}$ since the sum and product of positive definite functions are also positive definite.

Axiom EF3

Consider Axiom EF3,which says that not both x and $-x$ are positive. Suppose both x and $-x$ are positive definite members of $\mathcal{A}$. Then for each $\alpha \in \Omega$ we have $x(\alpha) > 0$ and $-x(\alpha) > 0$, contradiction. Hence Axiom EF3 holds at $\mathcal{A}$.

Axiom EF4

Consider Axiom EF4, which says that if both x and $-x$ are not positive, then x is zero. Suppose both x and $-x$ are satisfied at $\mathcal{A}$ to be not positive. That means that for every node $\mathcal{A}_\alpha$, x and $-x$ are not positive at $\mathcal{A}_\alpha$. That means that for every $\alpha \in \Omega$ $x(\alpha) \leq 0$ and $-x(\alpha) \leq 0$. Hence, $x(\alpha) = 0$. But as shown above, that implies x is identically zero. Hence $\mathcal{A}$ satisfies Axiom EF4.

Axiom EF5

Consider Axiom EF5, which says that if $-x$ is not positive, then x has a square root. If x is identically zero there is nothing to prove, so we may assume that x is not identically zero. If $\mathcal{A}$ satisfies that $-x$ is not positive, that means that $-x$ is not positive in any $\mathcal{A}_\alpha$; that is, $-x(\alpha) \leq 0$ for all $\alpha \in \Omega$. Then $x(\alpha) \geq 0$. Since this is true for every $\alpha \in \Omega$, and since Ω is dense in $\mathbb{R}$, and x is continuous, it follows that x is positive semidefinite. Hence $\sqrt{x(\sigma)}$ belongs to $\mathcal{A}$, by construction of $\mathcal{A}$. Hence $\mathcal{A}$ satisfies Axiom EF5.

Markov's principle

Suppose that $\neg\neg P(x)$ is satisfied at the root node $\mathcal{A}$.

Then for every α in Ω, $P(x)$ is satisfied at the leaf node $\mathcal{A}_\alpha$; that means that $x(\alpha) > 0$ for each α in Ω.

As shown in the verification of E5, this implies that x is positive semidefinite; and it is not identically zero since $x(\alpha) > 0$. Hence $P(x)$ is satisfied at the root node $\mathcal{A}$.

Positive elements have reciprocals

Now consider the axiom EF0, which says positive elements have reciprocals. Suppose x is positive at $\mathcal{A}$. Then x is positive definite. Hence $1/x$ belongs to C_{n+1}, where n is such that $x \in C_n$. Hence $1/x$ belongs to $\mathcal{A}$. Hence $\mathcal{A}$ satisfies EF0.

Not all nonzero elements have reciprocals

Consider the element of $\mathcal{A}$ given by the identity function, $i(t) = t$. Suppose $\mathcal{A}$ satisfies $i \cdot y = 1$, where 1 is the constant function with value 1. Then for each real number t we have $ty(t) = 1$. But this is a contradiction when $t = 0$. Hence $\mathcal{A}$ does not satisfy EF1.

Playfair does not imply Euclid 5

We obtain this result relative to the rest of the axioms of IECG. The field-theoretic version is that "elements without reciprocals are zero" does not imply "positive elements have reciprocals." It suffices to give a Kripke model satisfying "elements without reciprocals are zero" but not "positive elements have reciprocals." To say that $\neg\exists y\,(y \cdot x = 1)$ holds at a node $\mathcal{A}$ of a Kripke model is to say that no node above $\mathcal{A}$ contains an inverse of x. If one of the leaf nodes above $\mathcal{A}$ is a (classical) field, then x must be zero in that field, and hence in $\mathcal{A}$ also. Hence the axiom that elements without reciprocals are zero will hold in any Kripke model, all of whose leaf nodes are fields. What we need, then, is a Kripke model in which all the leaf nodes are ordered fields, and the root node has a positive element without a reciprocal.

A Kripke model satisfying Playfair but not Euclid 5

We construct a model similar to the one in the preceding proof, except that when constructing $\mathcal{A}$, we throw in only square roots, not reciprocals. More precisely, Let C_0 be the ring of polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ with coefficients in $\mathbb{K}$. For each nonnegative integer n, we define the ring C_{n+1} to be the least ring of real-valued functions containing C_n together with all square roots of positive semidefinite members of $\mathcal{C}_n$. Then the union $\mathcal{A}$ of the C_n contains square roots of its positive semidefinite members, but is not guaranteed by definition to contain their reciprocals.

EF2 through EF5

As in the previous proof, all the members of $\mathcal{A}$ have Pusieux series, so there is a countable set including all their zeroes; let Ω be the complement of this countable set, and define a Kripke model as before, with index set $\{0, \Omega\}$ and $R_0 = \mathcal{A}$ and for $\sigma \in \Omega$, $R_\sigma = C_\sigma$. The verifications of EF2 through EF5 are as in the previous proof.

Playfair holds in this model

Let x be an element of $\mathcal{A}$ such that $\forall y(xy \neq 1)$ holds at the root node $\mathcal{A}$. Then for all α in Ω and all y in $\mathcal{A}_\alpha$ we have $xy \neq 1$ in $\mathcal{A}_\alpha$, i.e. $1/x$ does not belong to A_α. But since A_α, as a field, is the quotient field of $\mathcal{A}$, this implies that x is identically zero.

Euclid 5 fails in this model

If Euclid 5 were satisfied at the root node, then every positive definite element of $\mathcal{A}$ would have its reciprocal in $\mathcal{A}$. To refute this, we must exhibit a positive definite element of $\mathcal{A}$ whose reciprocal is not in $\mathcal{A}$. For example, let $f(x) = x^2 + 1$. We do not even need to use square roots; the reciprocal function $1/(x^2 + 1)$ will never be generated, starting from polynomials and closing under square root and polynomial operations. Of course that claim requires a proof.

Proof of the claim (sketch)

Every function in $\mathcal{A}$ extends by analytic continuation to be a multi-valued function defined in the whole complex plane. This works, by induction following the construction of $\mathcal{A}$, because we only threw in square roots, not reciprocals.

To finish the proof, we show that $1/(x^2+1)$ is not in $\mathcal{A}$. The reason is simple: it has a unique analytic continuation into the complex plane, and that continuation has singularities at $x = \pm i$. Hence it cannot be continued to be a multi-valued function defined on $\mathcal{C}$. Hence it is not in $\mathcal{A}$. That completes the proof.

Independence of Markov's principle

We can define $\mathcal{A}$ to be the least ring of functions containing the polynomials and closed under square roots of positive semidefinite functions and reciprocals of nonzero functions. Interpret $P(x)$ at the root to mean "x is positive definite" instead of "x is positive semidefinite and not identically zero."

Now consider $x(t) = t^2$. Since 0 is not in Ω, $P(x)$ holds at each leaf node, since $x(\alpha) > 0$. Hence $\neg\neg P(x)$ holds at the root node. But $P(x)$ does not hold at the root, since x is not positive definite. The other field axioms do hold, as in the previous models.

Conclusions

- IECG is a constructive theory close in spirit and power to Euclid
- Constructive geometry differs from Euclid (only) in taking I.2 as an axiom and in strengthening the parallel postulate.
- Constructive logic is closely related to ruler-and-compass constructions.
- One can prove independence results for three versions of the parallel postulates
- These results nicely confirm Max Dehn's distinction between postulates and axioms