Formalization of Geometry

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Euclid, the first formalizer

- Museum at Alexandria, 350 BCE
- Research institute, university, think tank, DARPA
- library, lecture halls, residences
- pure math but also weapons research (catapults)
- scholars ate in common dining halls, lived on campus, held property in common.

- The library was accidentally burned, in part, by Julius Caesar in 48 BCE, but historian Strabo visited it, in working order, in 20 BC.
- Rome continued to fund it, especially arms research, but there is no evidence of the library after 275.

Plan of this lecture

- History of geometry
- History of formal logic (the two have intertwining roots)
- Axioms of geometry
- Comparison of work of Euclid, Hilbert, and Tarski: pencil-and-paper formalization
- Application of theorem-provers and proof-checkers to these three in the 21st century

Euclid's *Elements*

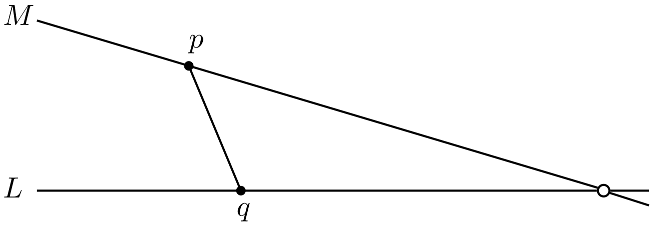
- Systematic (axioms and proofs) development of some mathematics.
- More advanced mathematics was already known, so maybe "Elements" meant "elementary", but not everything in the *Elements* is elementary.
- Book I is triangles, parallelograms, culminating in the Pythagorean theorem.
- Book II is about "equal figures" (in area)
- Greeks did not have algebra except as expressed in geometry.
- Book III is about circles.
- Book IV is about inscribed and circumscribed polygons.
- Book V is Eudoxes's theory of ratios, forerunner of Dedekind cuts.
- Book VI is about similar triangles
- Books VII-X are number theory
- Books XI-XII are about geometry in three dimensions, including the Platonic solids

Was Euclid an editor or an author?

- Some of the theorems in Euclid were already old, e.g., Thale's theorem (angle inscribed in a semicircle is a right angle) and the Pythagorean theorem; and Eudoxes's theory of ratios.
- But this is the first known axiomatic development (of any subject!), so Euclid was at least a systematizer.
- In an attempt to organize mathematical knowledge, it may have become apparent that certain chains of reasoning were circular, giving rise to the desire to sort out what depended on what.
- I speculate that the *Elements* began as course notes.

Euclid's Postulate 5, "parallel postulate"

If the two angles on the right of pq make less than two right angles together then line M meets line L.



Criticism of Euclid 5

- Already began with Proclus 450 CE
- Reached a crescendo in the 19th century
- People felt it to be less obvious than the other axioms, because the intersection point might be arbitrarily far away.
- Many famous people gave many incorrect “proofs”.
- Eventually it was realized that Euclid 5 is unprovable. There is such a thing as non-Euclidean geometry.
- You can read these stories in Greenberg’s book *Euclidean and Non-Euclidean geometries*.

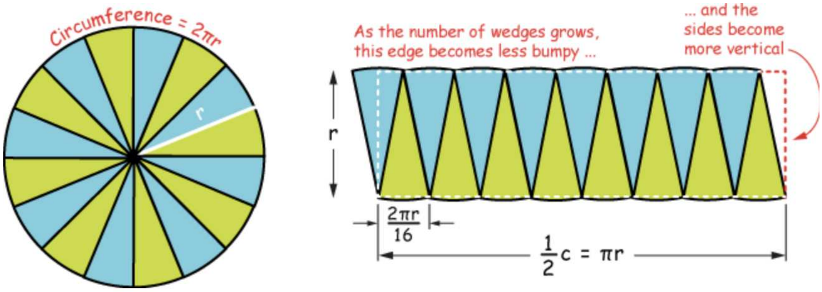
Increasing rigor

- The efforts to prove Euclid 5 led to more careful axiomatic studies of geometry.
- Euclid mentioned "same side" but never defined it.
- Euclid never even mentioned anything about the order of points on a line, but assumed that points were where they appeared to be in the diagram.
- Euclid assumed that circles that appear to intersect, do intersect.
- Pasch (1882) introduced "betweenness": $\mathbf{B}(a, b, c)$ if abc occur in that order on a line.
- Pasch also introduced "Pasch's axiom", which we will examine later.

Area

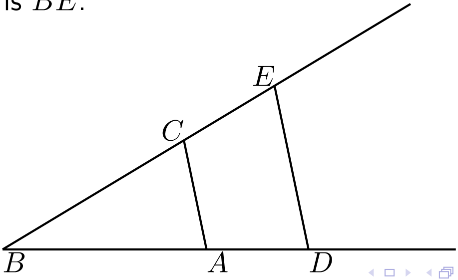
- Euclid never even mentions the word "area".
- He probably realized he did not know how to define it.
- The Greeks apparently did not, in their mathematics, acknowledge that one measures area with a number, although of course engineers and architects must have done so.
- Even when Archimedes worked out the area of a circle, he stated his result by saying that a circle is equal to a rectangle whose height is the radius and whose width is half the diameter. He didn't mention area!
- Euclid said two figures "are equal" rather than "have equal area".

Archimedes found a rectangle equal to a circle

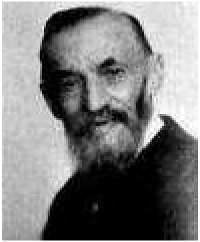


Defining arithmetic in geometry

- To study similar triangles and proportion, you need something like multiplication. But the Greeks never realized that multiplication of line segments can be geometrically defined.
- Descartes (1637):
- Let AB be taken as unity, and let it be required to multiply BD by BC.
- The answer is BE.

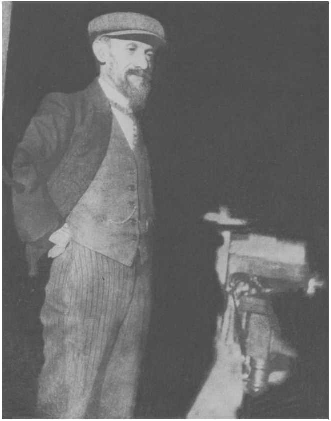


Guiseppe Peano, the father of modern logic 1858-1932



- Introduced the logical symbols we use today
- Led a group of mathematicians whose aim was to write formal proofs of all mathematics.
- Founded a journal in which such papers were printed.
  - His famous theorem on existence of solutions of $y' = f(x, y)$, when f is continuous, was published in symbols-only form, and only became known years later when someone "translated" it into German and published it in a German journal.
  - Geometry was for him just an example of a theory to formalize.
  - Nevertheless he made an important contribution: "inner Pasch" (which we will examine later)

Peano and his printing press



Peano had troubles with the printer and typesetter, so he bought his own printing press to print his journal.

Roots of modern logic in the 19^{th} century

- George Boole, 1852, invented Boolean algebra. Used algebraic symbols and dealt with propositional logic only.
- Pasch, 1882, rigorous treatment of geometry.
- Frege, 1893, Introduced theories of quantification and classes. His two-dimensional notation was awkward and impossible to typeset, so his notation did not survive. His theory contained a paradox, pointed out in a famous letter from Russell. Nevertheless it was influential.
- Peano, contemporary of Frege, but worked independently.

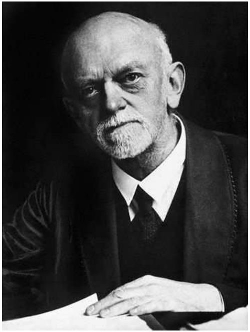
Roots of type theory

- Russell and Whitehead wrote *Principia Mathematica* (PM), working out the details of type theory.
- Several of today's theorem provers are based on Higher Order Logic, which is a direct offshoot of Russell's type theory.
- Russell and Whitehead adopted Peano's symbols and Frege's ideas about quantification and properties
- PM gave a well-worked out example of a formal system, and struggling with it led to the notion of first-order logic.

Roots of first-order logic

- First-order logic was largely due to Thoralf Skolem (1887-1963), who proved important theorems in the early 1920s. He published them mostly in Norwegian.
- The concepts of syntax and semantics, theory and model, were not quite clear before then; indeed the question of completeness was first posed in writing in Hilbert-Ackerman 1929.
- Soon after it was posed, Gödel answered it, and then went on to prove his incompleteness theorems, which still referred to PM, as what is now known as PA (Peano arithmetic) was not yet a standard theory.

Hilbert's *Foundations of Geometry*, 1899



- Hilbert was a strong proponent of the axiomatic method, by which he meant that for each branch of mathematics, one should write down some axioms, and derive all the theorems from those axioms.
- Geometry was for him "just an example", as it was with Peano.
  - But he devoted years of effort to geometry. He gave three academic-year courses interspersed with two summer-school lecture series, and these lectures were preserved and in recent years published, so we can see the development of his thought.
  - His final axiom system, in his 1899 book, is not quite a first-order theory–first order logic was not yet understood, so set theory and natural numbers are used freely.

Tarski's geometry

- first-order theory with 12 to 15 axioms
- one-sorted, variables only for points
- angles treated as triples of points
- lines treated as pairs of points
- Developed in the 1920s
- Manuscript at the printers destroyed by bombing
- Not published until 1958 and then without details
- Developed at Berkeley in 1960s by Gupta and Szmielew
- Finally published in SST = Szmielew, Schwäbhauser, Tarski (1983)

Summary

That completes the historical part of this lecture.
To recap, chronologically:

- Euclid 350 BCE
- Descartes 1637
- Boole 1851
- Pasch 1882
- Peano 1890
- Frege 1893
- Hilbert 1899
- Russell and Whitehead 1908
- Skolem 1920-28
- Hilbert-Ackermann 1928
- Gödel 1930-31
- Tarski 1927–1958

Euclid's axiomatic framework

- Definitions
- Common notions
- Axioms (universal statements)
- Postulates (constructions asserted to be successful)

Common Notions

- Things which are equal to the same thing are also equal to each other.
- If equals be added to equals, the wholes are equal.
- If equals be subtracted from equals, the remainders are equal.
- Things which coincide with one another are equal to each other.
- The whole is greater than the part

Adding and subtracting are used for lines, angles, and "figures." Equality and "coincidence" refer to congruence. In first-order logic we need both equality axioms and congruence axioms to provide the effects of Euclid's common notions. That still doesn't cover the applications to angles and figures.

Euclid's Five Postulates

- To draw a straight line from any point to any point.
- To produce [extend] a finite straight line continuously in a straight line. [How far? Euclid is vague on that point!]
- To describe [draw] a circle with any centre and distance [radius].
- That all right angles are equal to one another. (A right angle is one "set up on a straight line that makes the adjacent angles equal to one another.")
- That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Postulates Euclid omitted but ought not to have omitted.

- line-circle continuity. A line with one point inside a circle must meet the circle.
- circle-circle continuity. A circle with one point inside and one point outside another circle must meet that circle.
- SAS (side-angle-side implies triangle congruence). Euclid stated it as Prop. I.2, but even in antiquity his "proof by superposition" was rejected.
- Pasch's axiom, introduced in 1882: a line that meets one side of a triangle, and lying in the plane of the triangle, must meet one of the other two sides of the triangle.

All formalizations of geometry after Pasch used SAS and Pasch, and either the two continuity axioms mentioned, or yet-stronger continuity axioms.

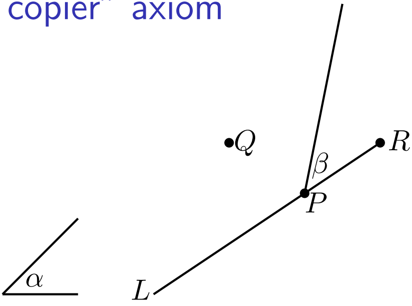
Hilbert's *Foundations of Geometry*

- Hilbert's 1899 book, the culmination of a decade of thought about both axiomatization and geometry.
- First-order logic was not yet developed. Hilbert's theory was not first-order.
- Hilbert had points, lines, and planes as primitive sorts, but line segments and rays were treated as sets of points.
- Angles were defined as pairs of distinct rays with a common vertex. Pairing seems to be primitive, i.e. not set-theoretically defined.
- Although angles were defined, angle congruence is taken as a primitive relation!
- The notions of "same side" and "opposite side" were taken as primitive. (Euclid mentioned them but did not define them or take them as primitive, causing problems with his proofs.)
- Hilbert had a complicated "continuity axiom" that was not first order, but we won't discuss it here .

Hilbert's axioms

- Hilbert's axioms were pretty strong, in the sense that a smaller set of axioms could work. He wanted to get somewhere fast, and was not interested in finding a minimum axiom set.
- Since points, lines, and planes are all primitive, he needed "incidence relations" (point lies line, point lies on plane). "Line lies on plane" can be defined then.
- There were "congruence axioms". One of them was SAS.
- Axioms of congruence asserted that congruence of line segments is an equivalence relation
- Axioms of order (on a line), stated in terms of betweenness $\mathbf{B}(a, b, c)$
- Pasch's axiom, described as a "plane axiom of order."

Hilbert's "angle copier" axiom



- Given an angle α and two points PR on a line L and a point Q not on L, you can find an angle β with vertex at P and R on one side of angle β, all of whose interior points lie on the same side of L as Q, such that $\beta = \alpha$. Here $=$ means angle congruence.
- Uniqueness of the copy β is specified as *part of the axiom*.
- Hence Euclid's "part not equal to the whole" for angles is built-in: an angle cannot be less than itself.

Hilbert's axioms, continued

- "same side" and "opposite side" defined only for points and lines in the same plane. For A and B not on L, A and B are on the opposite side of L if segment AB meets L and on the same side if AB does not meet L.
- Plane separation theorem: Given A and L, every point B in the plane of A and L is either on the same side of L as A or the opposite side.

Archimedes's axiom

Let a segment a and two points A and$B$ be given on line L. Then it is possible to define a number of points $A_1, \ldots, A_n$ such that B lies between A and A_n and each of the segments $A_1A_2, \ldots, A_{n-1}A_n$ is congruent to a.

- This is not a first-order axiom of geometry as it mentions "natural number" (via the three dots).
- So it would be better to not use it, if possible, and Hilbert did not include it.

What did Hilbert prove in his book?

- Pascal's theorem implies Euclid's theory of proportions
- Then we can define segment multiplication. (Addition is trivial.)
- Desargues's theorem implies associativity.
- Pascal's theorem implies commutativity.
- (The mentioned theorems are important in "projective geometry" and were well-known.)
- Thus arithmetic can be defined in geometry.

This is an important result, leading to the characterization of the models of geometry as being of the form $\mathbb{F}^2$ for certain kinds of fields $\mathbb{F}$. But Hilbert himself didn't prove such theorems. Instead, he was eager to get on (in the Appendix) to defining arithmetic somehow in non-Euclidean geometries.

Tarski's geometry

- Only one sort: points
- Lines are only referred to by two points. That's what Euclid does, too. You never see "line L" in Euclid, only AB.
- Angles are only referred to by three points, as "angle ABC". That's also what Euclid does.
- Angle congruence is a defined concept (unlike Hilbert).
- order for angles is also a defined concept, and $\alpha \not< \alpha$ is a difficult theorem.
- This economy of concepts permits an economy of axioms: Tarski started with 16 axioms and (decades later) had only 11, since the rest had been proved from the 11 by Tarski's students.
- And the axioms all have short statements without using definitions or abbreviations.

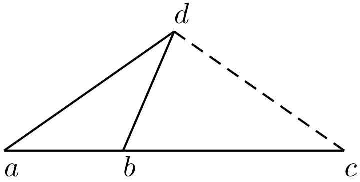
Betweenness

If B is between A and C, then do we allow $B = A$ or $B = C$, or is B required to be different from A and C?

- If B has to be different from A and C, that is *strict betweenness*
- Otherwise it is *non-strict betweenness*.
- Hilbert has strict betweenness
- Tarski has non-strict betweenness
- Either one can define the other one
- but it makes the axioms not directly comparable, a formal annoyance.
- In constructive geometry, I used strict betweenness, reformulating Tarski's axioms accordingly.

Tarski's 5-segment axiom

This is how one expresses the SAS congruence principle in Tarski's system.



If the four solid lines on the left are equal to the corresponding solid lines on the right, then the dashed lines are also equal.

This is SAS for $dbc = DBC$, interpreting the congruence of triangles abd and ABD as expressing the congruence of angles dbc and DBC.

History of the 5-segment axiom

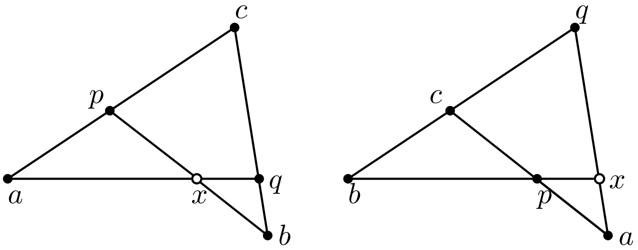
- The key idea (replacing reasoning about angles by reasoning about congruence of segments) was introduced (in 1904) by J. Mollerup.
- His system has an axiom closely related to the 5-line axiom, and easily proved equivalent. Tarski's version, however, is slightly simpler in formulation.
- Mollerup (without comment) gives a reference to Veronese (1891) Veronese does have a theorem (on page 241) with the same diagram as the 5-line axiom, and closely related, but he does not suggest an axiom related to this diagram.
- Hence Mollerup gets the credit.

Dimension axioms

- Euclid's Books I-III are about plane geometry, but it is meant to be "any plane".
- Euclid has no axiom saying that every point lies in a plane, and he can't have meant to assume that, since in later books he deals with three-dimensional space and the Platonic solids.
- Hilbert's theory is about three-space. He has an axiom that there are four points not lying on any plane, for example. That is a "lower dimension axiom".
- Hilbert has an upper dimension axiom too: if two planes have a point in common then they have another point in common.
- Tarski states a lower and upper dimension axiom for each n, but wishes to develop as much as possible dimension-free.

Pasch's axiom in Tarski's theory

- Pasch's version of Pasch's axiom, that a line that meets one side of triangle ABC must meet another side, doesn't work in 3-space without requiring explicitly that the line lie in the plane of ABC.
- This defect was remedied by Peano in 1890, who invented the axioms now known as *inner Pasch* and *outer Pasch*.



Inner Pasch (left) and outer Pasch (right). Line pb meets triangle acq in one side ac, and meets an extension of side cq. Then it also meets the third side aq. The open circles show the points asserted to exist.

Relations between inner and outer Pasch

- In formalizing Euclid, I needed both, so I took both as axioms.
- Tarski originally took both. But Gupta's thesis (1965) showed that either one implies the other one.
- So the choice was an aesthetic one.
- Tarski preferred outer Pasch, but Szmielew chose inner Pasch and that was used in SST.
- The proofs of the two implications are both difficult, but perhaps inner implies outer is a bit more difficult.

Degeneracies in inner and outer Pasch

- with non-strict betweenness, these axioms include degenerate cases, even when all the points collapse to lie on a single line.
- That was fine with Tarski; these degenerate cases enabled his students to prove some betweenness axioms that were originally thought to be needed.
- If you want the minimal axiom system and do not care about the axioms expressing geometrical intuitions, that's fine.
- On the other hand, if you do care about the axioms expressing intuitions, and not so much about whether you have 12 or 16 axioms, use strict betweenness and put back the original betweenness axioms.
- You *have to* do that if you want to do constructive geometry.

Extension axiom

- Euclid said a line can be "produced", but didn't say how far.
- Hilbert said, if AB is any line, and P any point, then there is a point Q on a given ray with vertex Q such that $PQ = AB$ [Axiom III,1, misleadingly translated in the Open Court edition].
- Tarski's version is the same as Hilbert's: segment RP can be extended past P by amount AB.
- Since Tarski uses non-strict betweenness, the cases $Q = P$ and $A = B$ are both allowed. The axiom with strict betweenness seems better to express the intuition involved.

Rigid and collapsible compass, and Euclid I.2

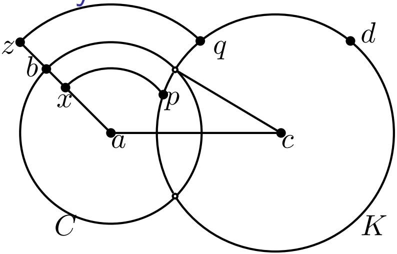
- A *rigid compass* allows you to set the compass to AB, then pick it up and transfer the distance to point P, constructing the extension of RP as postulated by Hilbert and Tarski.
- A *collapsible compass* does not allow you to do that. You can only draw a circle with center P and passing through an existing point X.
- Euclid's compass was collapsible, as given by his circle-construction postulate. And his segment-extension postulate doesn't specify how far you can extend RP.
- Thus Euclid needs to *prove* Hilbert and Tarski's extension axiom, which he does in Prop. I.2.
- That is, "a collapsible compass can simulate a rigid compass."
- I.2 is a beautiful proof, and it is a shame to make it a one-liner by choosing an axiom that includes it.

Line-circle continuity

There are several possible versions of an axiom of that name.

- A point P is *inside* circle C with center A if there is a point B on C and a point Q between A and B with $AP = AQ$.
- P is *outside* circle C with center A if there is a point B on C and a point Q with $\mathbf{B}(A, B, Q)$ and $AP = AQ$.
- Segment-circle continuity: If P is inside circle C and Q is outside, then PQ meets circle C.
- Line-circle continuity: if P is inside circle C then any line L through P meets C.
- Or we could require L to meet C twice, with P between the two points where L meets C.
- The proofs of equivalence of these axioms are not quite trivial.

Circle-circle continuity



- If circle K has a point p inside circle C and a point q outside K, then C meets K.
- Or maybe, C meets K twice.
- Once is enough for Euclid I.1 and I.22, and twice doesn't help with improving the proof of I.1.
- Axiomatically it doesn't matter, because:
- line-circle implies circle-circle and vice versa, but the proofs are difficult.

Tarski's continuity axiom

- Tarski's continuity axiom (A11) is a schema. It says that if a Dedekind cut is defined by a first-order formula, it is filled.
- I used to think A11 easily implied line-circle and circle-circle–until I tried to prove it.
- The reason it is difficult will be discussed soon

Gupta's amazing 1965 Ph. D. thesis

H. N. Gupta's thesis, written under Tarski at UC Berkeley (1965) contained these results, all proved without using any continuity axioms, including line-circle and circle-circle and A11.

- Inner Pasch implies outer Pasch.
- Outer Pasch implies inner Pasch.
- The base of every isosceles triangle has a midpoint.
- Every segment has a midpoint.
- Dropped perpendiculars exist, i.e. there is a perpendicular to line L from point P not on L.
- Erected perpendiculars exist, i.e. there is a perpendicular to line L at a point P on L.
- Other difficult results about betweenness axioms.

The constructions and long chains of reasoning in these proofs are beautiful and amazing. Imagine: construct perpendiculars without using any circles!

What was done with and in Tarski's theory

- Up until 1965, Tarski's students worked on reducing the number of axioms.
- Szmielew and possibly Tarski worked out proofs of many theorems, using Gupta's results to provide perpendiculars and midpoints.
- They followed Hilbert, struggling to prove Pappus's and Desargues's theorems and define addition and multiplication.
- They succeeded in these efforts; as a consequence, they proved all of Hilbert's axioms from Tarski's.
- These results were in Szmielew's manuscript about 1965, but not published until 1983 in SST: Szmielew, Schwäbhauser, and Tarski.
- Part I of SST is due to Szmielew "with inessential modification", and is the only place where Gupta's work was ever published.
- *But they never went back to prove Euclid's theorems!*

To prove A11 or segment-circle implies line-circle

- Given line L with a point P inside circle C. To prove L meets C, either by A11 or circle-segment, we will need a point Q on L outside C.
- Obvious ways to construct Q need dropped perpendiculars and the theorem that the hypotenuse of a right triangle is greater than the side.
- But to get dropped perpendiculars without using line-circle, we need Gupta's very difficult results.
- Well, it works, but it is not trivial if it relies on Gupta.
- I acknowledge Richter for pointing this out.

What was Euclid thinking?

- It seems strange that Euclid, who was generally careful, glaringly omits both line-circle and circle-circle.
- When he needs to use line-circle in the proof of I.2, he instead says "Let the straight line AE be produced in a straight line with DA". In other words, "let DA be extended until it meets the circle at E."
- Lines are always finite, so line-circle intuitively says that a line can be extended until it meets the circle, as well as saying that (when it is long enough to reach the circle) it cannot pass through the circle at some "missing point" without touching.
- Probably Euclid thought the difficulty was getting the lines long enough, not getting the circle impenetrable.
- Then he probably had line-circle in mind when stating Euclid 2, "To produce a finite straight line continuously in a straight line", not just "by some amount", and not "by an amount equal to a given segment", but "enough to meet a given circle", if the starting point is inside that circle.

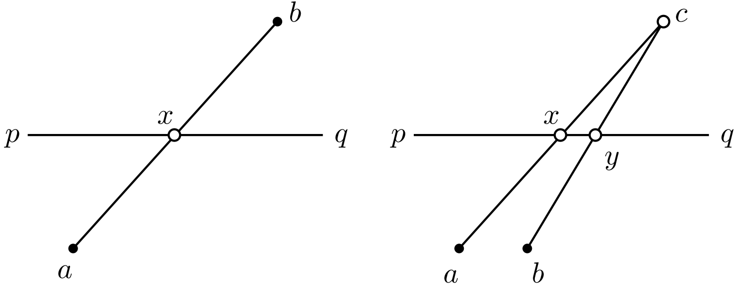
Betweenness axioms

- Here are the betweenness axioms we used in formalizing Euclid.
- It makes a big difference whether $\mathbf{B}$ is strict or non-strict betweenness, and whether inner Pasch (or outer Pasch) is taken with strict or non-strict betweenness.
- We state the axioms using strict betweenness

$$\begin{array}{rl}
\text{identity} & \neg\mathbf{B}(a,b,a) \\
\text{symmetry} & \mathbf{B}(a,b,c) \ \rightarrow\ \mathbf{B}(c,b,a) \\
\text{inner transitivity} & \mathbf{B}(a,b,d) \wedge \mathbf{B}(b,c,d) \ \rightarrow\ \mathbf{B}(a,b,c) \\
\text{connectivity} & \mathbf{B}(a,b,d) \wedge \mathbf{B}(a,c,d) \ \wedge \\
 & \neg\mathbf{B}(a,b,c) \wedge \neg\mathbf{B}(a,b,c) \ \rightarrow\ b = c
\end{array}$$

Same side and opposite side according to Tarski

- Tarski gave definitions of "same side" and "opposite side" that work even without a dimension axiom.
- These definitions can be used to define *plane*. The plane determined by a line L and point P consists of all the points that are on L, or on the opposite side of L from P, or on the same side of L as P.



(Left) a and b are on the opposite side of pq. (Right) a and b are on the same side of pq if there exist points x and y collinear with pq, and a point c, such that $\mathbf{B}(a, x, c)$ and $\mathbf{B}(b, y, c)$.

What about Euclid's Postulate 4?

Euclid's Postulate 4 says "all right angles are equal."

- There is a long history of claims that Postulate 4 is provable.
- It matters a lot what the other axioms are.

History of “proofs” of Postulate 4

- Proclus §189 offers a proof, using angle-copying and trichotomy for angles, and “the whole is greater than the part” for angles, and assumes everything is in one plane.
- Proclus’s proof works using Hilbert’s axioms, see Prop. 3.23 of Greenberg’s text, 4th edition.
- Proclus taught in Athens, and died in 485. Thus Euclid’s *Elements* were already 700 years old.

Proving Postulate 4 from Tarski's axioms

- done in SST, much harder than in Hilbert.
- One proves that congruence is preserved under point reflections and reflections in a line and translations and rotations.
- That reduces it to the case where the two right angles have the same vertex, and one side in common.
- but they could still be in different planes, so we are far from done.

Different versions of the parallel postulate

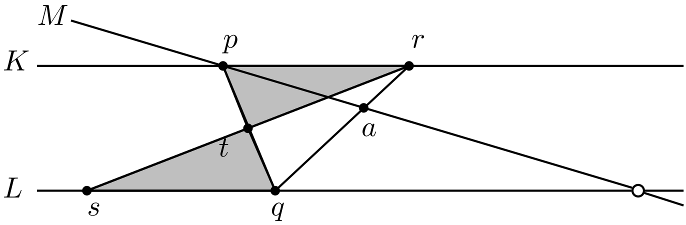
There are many propositions equivalent to the parallel postulate (in "neutral geometry", i.e. without any parallel postulate). Here are a few:

- Euclid's Postulate 5
- Playfair's postulate
- Triangle circumscription
- Tarski's version
- There are at least 34 propositions equivalent to the parallel postulate

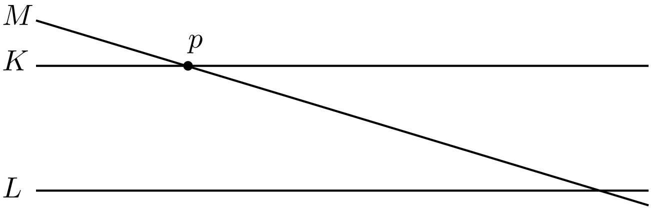
I will show you pictures of the ones named here.

Euclid 5

Here is a points-only version. (It doesn't mention angles.) The hypothesis is that the gray triangles are congruent and $\mathbf{B}(q, a, r)$.

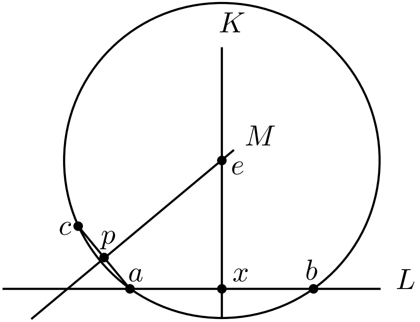


Playfair's postulate



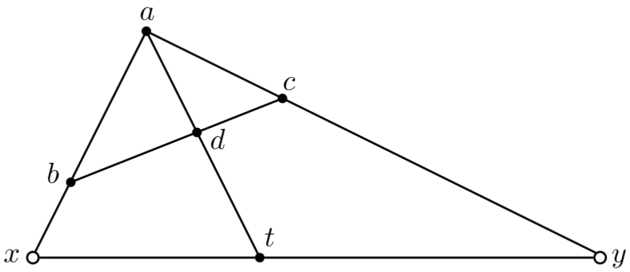
- There can't be two parallel lines to L through p.
- No existential assertion at all.
- This is the version Hilbert used.

Triangle circumscription



- Any three non-collinear points lie on a circle.
- Equivalently, for any three non-collinear points, there exists a fourth point equidistant from all three (the center).
- Euclid IV.5 proves the triangle circumscription principle.
- The converse implication was first proved by Farkas Bolyai, father of Janos Bolyai, who thought he had proved Euclid's parallel postulate, but had actually assumed the triangle circumscription principle. See Greenberg, pp. 229–30, p. 240.

Tarski's parallel axiom



- If t is in the interior of angle abc then any line through t meets the sides of the angle.
- Mentioning point d allows avoiding mention of angles.

1950s

- Early computers had just come into existence.
- Already people began trying to make them prove theorems.
- They started with propositional logic and geometry.
- Gelernter's *Geometry Machine* 1959 ran on the IBM 704
- It had 20,000 instructions in a special-purpose list-processing language which was then compiled into FORTRAN.
- It found solutions to fifty problems taken from high-school textbooks and final examinations.
- running times up to 30 minutes.

Computational analytic geometry

- Analytic geometry: write everything in (x, y) coordinates and use algebra.
- You can use a computer to do the calculations and thus prove geometry theorems.
- The algebra can be tricky. Several methods were used:
- Wu's method
- Chou's method
- The area method
- Gröbner bases
- Only works for theorems that translate as equations (not inequalities)
- None of these qualifies as "formalization" because (unverified) computation is used
- The list of people and papers involved would be several slides long; for history see my ADG 2012 paper.

Tarski's system in OTTER

- Art Quaife's 1990 book had a chapter on Peano Arithmetic and a chapter on Tarski's geometry.
- Wos's early work with Tarski geometry (1988 and 2003)
- Qaife and Wos left some unsolved "challenge problems."
- Wos and Beeson returned to these problems in 2010
- We proved every theorem in 12 chapters of SST, one at a time, giving the prover the previous theorems to work from.
- Over 200 theorems altogether.
- We used various "strategies" to help OTTER find the proof.
- Sometimes it was more like a proof-checker than a prover.
- But we solved all the "challenge problems" left unsolved by Quaife and Wos.

Hilbert in HOL Light

- Richter formalized Hilbert in HOL Light
- His work is distributed with HOL Light as `RichterHilbertAxiomGeometry`
- two-dimensional only
- Proved theorems in absolute geometry including most of Euclid I.13 to I.28

The Coq group at Strasbourg

- Julien Narboux at Strasbourg, France started this project in 2012.
- Gabriel Braun, Pascal Schreck, and by 2014, Pierre Boutry were involved.
- Coq can be used to define algorithms that are then automatically proved correct.
- They formalized Wu's method and the area method
- Then they started to formalize Tarski geometry, getting several chapters into SST.

ADG 2012

The dragon, as in maps of old, represents uncharted and possibly dangerous territory.

Geometric Theorem ⟶ **Algebraic Translation**

Chou, Wu, Descartes

Euclid
Tarski
Szmielew
Narboux

Wu-Ritt method

Chou's area method

CAD (Collins)

Gröbner bases

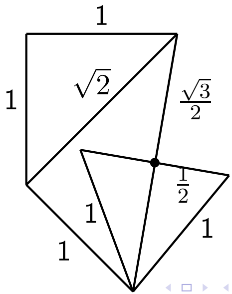
Descartes, Hilbert

Geometric Proof ⟵ **Algebraic "Proof"**

I proposed at ADG 2012 to go around the diagram from upper left clockwise to the lower left.

The Coq group at Strasbourg did it!

- They formalized the definition of multiplication in Tarski's theory.
- Then they could adapt their earlier work on analytic geometry to get first-order proofs in Tarski's theory
- They answered a challenge in my ADG paper: construct an equilateral triangle with given base, without using circles. (The construction is Hilbert's.)



Formalizing Euclid

- Hilbert and Tarski took a different path than Euclid
- They did not use circles, and instead focusedon segment arithmetic
- Euclid needs some corrections!
- Euclid Book I formalized in 2012 by Beeson, Wiedijk, and Narboux
- proofs written in a fragment of first-order logic
- checked first with a custom proof-checker or “proof debugger”
- Translated into HOL Light and Coq and checked in both checkers.

Summary

- Formal geometry has roots in Euclid.
- Efforts to improve the rigor of Euclid have been ongoing for 2370 years.
- In the nineteenth century that was part of a larger struggle to improve rigor in mathematics.
- Computers were applied to proving theorems from the outset, and geometry was always of interest.
- The old split between analytic and synthetic cropped up again in computer geometry
- Now Tarski and Hilbert, and Euclid Book I, have all been computer-verified.

Reading list

- Euclid's *Elements*. Get the Green Lion Press edition, all thirteen books in one volume.
- Greenberg, *Euclidean and non-Euclidean geometries*, 4th edition
- Schwabhäuser, Szmielew, and Tarski. (German; I can give you a free copy.)
- *Proof-checking Euclid*, by Beeson, Narboux, and Wiedijk. Annals of Mathematics and Artificial Intelligence, 85(2): 213-257. January 2019.
- A. Tarski and S. Givant, *Tarskis system of geometry*, The Bulletin of Symbolic Logic, 5 (1999), pp. 175214.
- Hilbert, *Foundations of Geometry*.
- *Finding Proofs in Tarskian Geometry*. Beeson, M.; and Wos, L. Journal of Automated Reasoning, 58(1): 181-207. 2017.
- All my papers are listed at `http://www.michaelbeeson.com/research/papers/pubs.html`. If you're reading this online you may be able to click on that URL successfully.