Triangle Tiling

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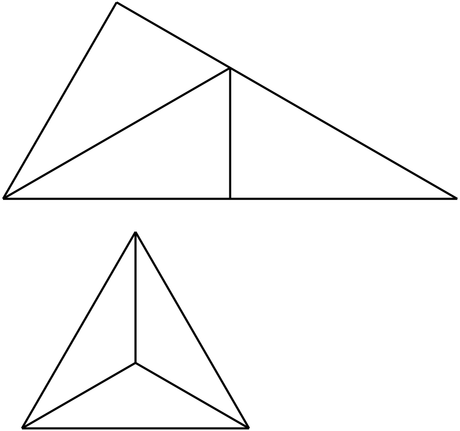
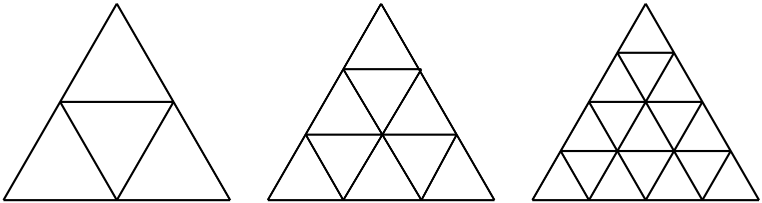
Two 3-tilings

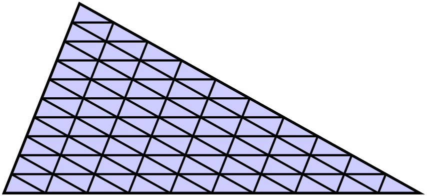


A 4-tiling, a 9-tiling, and a 16-tiling



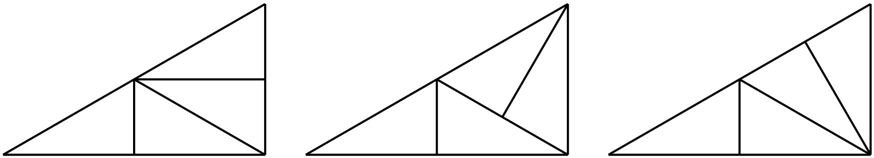
This illustrates the *quadratic tilings*, possible when N is a square, for any triangle.

Quadratic tiling of any triangle

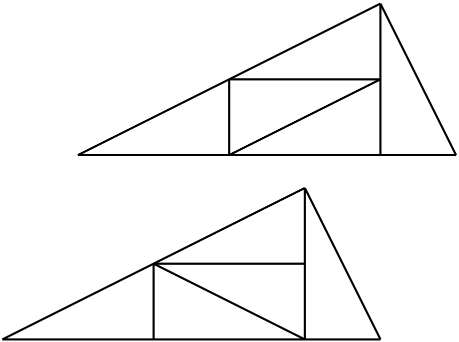


N is a square.

Three 4-tilings

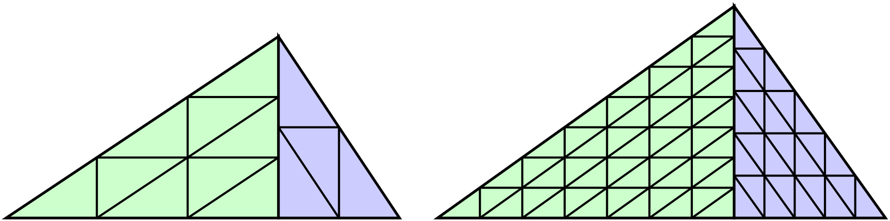


Two 5-tilings

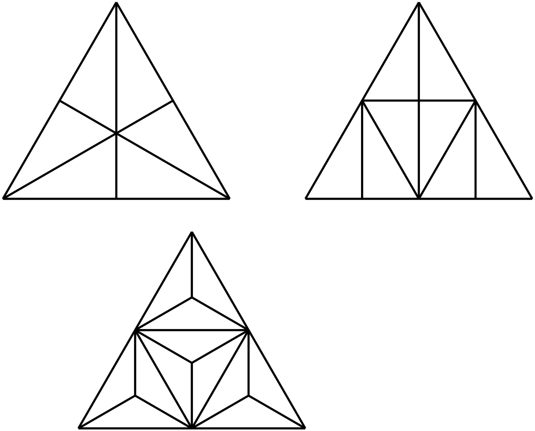


Biquadratic tilings with $N = e^2 + f^2$

Here we have $N = 13 = 3^2 + 2^2$ and $N = 74 = 5^2 + 7^2$.



A 6-tiling, an 8-tiling, and a 12-tiling



Definition of N-tiling

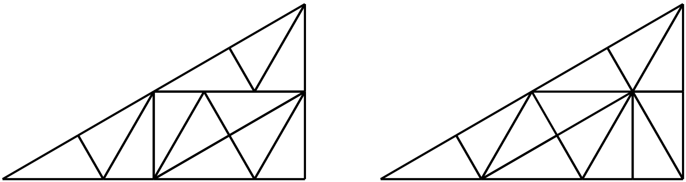
An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile".

The tile may or may not be similar to ABC. We wish to understand possible tilings by completely characterizing the triples (ABC, T, N) such that ABC can be N-tiled by T. In particular this understanding should enable us to

- Given N, find T and ABC (or say no tiling is possible).
- Given ABC, find T and N (if possible).
- Describe the set of N such that some triangle can be N-tiled by some tile.

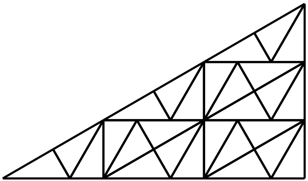
The $3m^2$ tilings

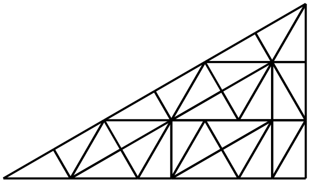
$N = 3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. Here is the case $m = 2$ and $N = 12$:



The $3m^2$ tilings

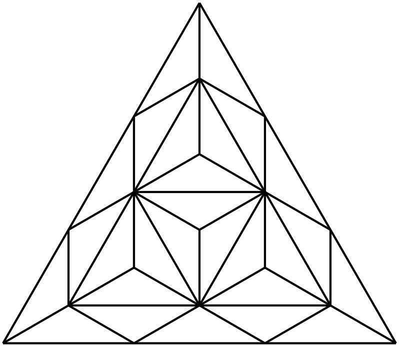
Here is the case $m = 3$, $N = 27$:





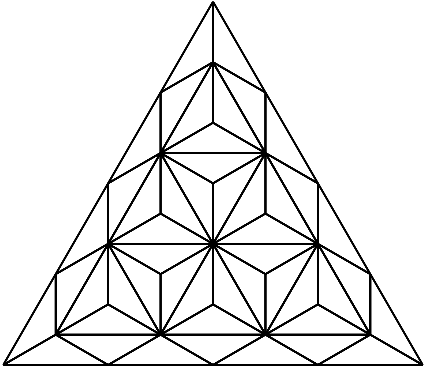
Until October 12, 2008, no examples were known [to me] of more complicated tilings than those illustrated above.

A prime 27-tiling



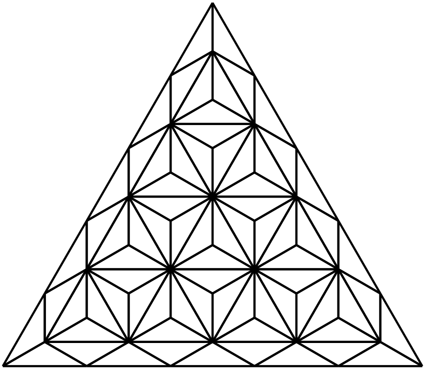
Then I found a family of $3m^2$ tilings, built from hexagons.
In 2012 I found that perhaps this one was known to Major MacMahon long ago (1921).

$3m^2$ tilings for $m = 4$, $N = 48$



This tiling is made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides.

$3m^2$ tiling for $m = 5$, $N = 75$



In general one can arrange $1 + 2 + \ldots + k$ hexagons in bowling-pin fashion, and add $k + 1$ tiles on each of three sides, for a total number of tiles $3(k + 1)^2$.

Two $25 Erdős Problems

- Find all positive integers N such that at least one triangle can be cut into N triangles congruent to each other.
- Find (and classify) all triangles that can only be cut into n^2 congruent triangles for any integer n.

These problems are mentioned in Soifer's book, *How does one cut a triangle?*, p. 48.
Both are still unsolved.
I say that Erdős undervalued these problems.

A new theorem

The conjecture that originally interested me in this subject is now a theorem:

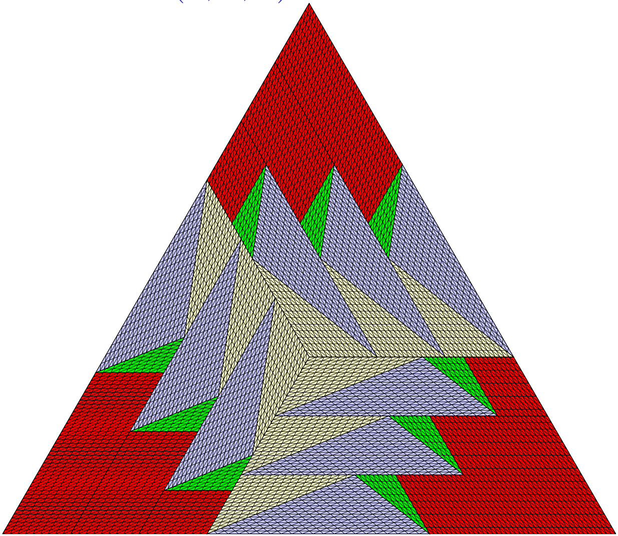
There is no 7-tiling (of any ABC by any tile).

Moreover, there is also no 11-tiling. But I still do not know if there is or is not a 19-tiling. One might hope that if p is a prime of the form $4n + 3$ then there is no p-tiling. But I do not even dare to label that a conjecture.

Another new theorem

- If ABC is an equilateral triangle, and there is an N-tiling of ABC (and $N > 3$), then N is not a prime number.
- Until December, 2018, it was not known even if there are arbitrarily large N for which no equilateral triangle can be N-tiled.
- There are tilings of equilateral ABC by lots of different tiles, but N might be very large.
- Nothing much is known about the sequence of possible N, except that it doesn't contain any primes greater than 3.

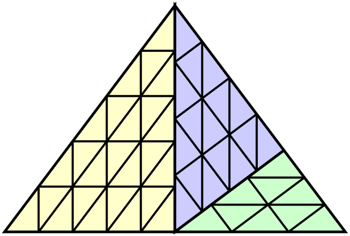
$N = 10395$ with tile $(3, 5, 7)$



Is there an N-tiling of an equilateral triangle for any N between 11 and 10395?
That's a big gap, but we don't know the answer.
We know some N that won't work (primes, for example), but about others we don't know.

$\mathbf{d}$-matrix

The sides are sums of edges of tiles.
The $\mathbf{d}$ matrix puts those equations in tabular form.



$$\begin{pmatrix} 25 \\ 25 \\ 30 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 5 \\ 3 & 4 & 0 \\ 5 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Eigenvalues of $\mathbf{d}$-matrix

If ABC is similar to the tile, then the lengths of the sides of ABC are $\sqrt{N}$ times (a, b, c).

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

That is, $\sqrt{N}$ is an eigenvalue of $\mathbf{d}$.

Then one can bring to bear the theory of eigenvalues and eigenvectors, and eventually prove Snover's theorem (1991)

Theorem (Snover *et. al.*)

Suppose ABC is N-tiled by tile T similar to ABC. If N is not a square, then T and ABC are right triangles. Then either

(i) N is three times a square and T is a 30-60-90 triangle, or
(ii) N is a sum of two squares $e^2 + f^2$, the right angle of ABC is split by the tiling, and the acute angles of ABC have rational tangents e/f and f/e,

and these two alternatives are mutually exclusive.

Miklós Laczkovich

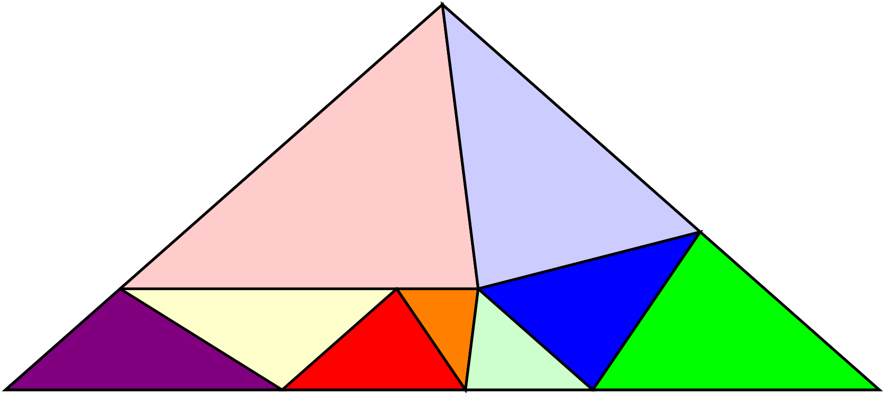
Miklós Laczkovich has written many papers on triangle tiling, starting in 1995 and continuing to the present day. He is a retired professor living in Budapest. He has also written uncountably many papers on other subjects.

When the angles of ABC are all rational multiples of π

Then the coordinates of all the vertices in the tiling belong to some cyclotomic field $\mathbb{Q}(\zeta)$, with $\zeta = e^{2\pi i/k}$.
That field has well-known automorphisms. Those automorphisms preserve lines and angles (but not lengths). So they take triangles into similar triangles, but not necessarily congruent triangles.
Therefore a tiling of ABC goes into a tiling of some other triangle, but only by *similar* triangles, not congruent ones. If one angle in tiling is composed of two smaller angles, then the automorphism must preserve that sum. Then one works with those equations and eventually determines the possible tilings (after a lot of calculations).
Laczkovich carried out this work in 1990 and 1995.

Dissection of ABC into triangles similar to $(4, 5, 6)$

You could cut these triangles all into smaller *congruent* triangles, but you would need 5,861,172 of them.



Laczkovich's table of 1995

Table : Laczkovich's 1995 list of tilings by tiles with commensurable angles

ABC	the tile
(α, β, γ)	similar to ABC
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
equilateral	$\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$
equilateral	$\left(\frac{\pi}{3}, \frac{\pi}{12}, \frac{7\pi}{12}\right)$
equilateral	$\left(\frac{\pi}{3}, \frac{\pi}{30}, \frac{19\pi}{30}\right)$
equilateral	$\left(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30}\right)$

Laczkovich 2012

Table : Laczkovich's 2012 improvements, with congruent tiles

ABC	the tile
(α, β, γ)	similar to ABC
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$
not possible	$(\frac{\pi}{3}, \frac{\pi}{12}, \frac{7\pi}{12})$
not possible	$(\frac{\pi}{3}, \frac{\pi}{30}, \frac{19\pi}{30})$
not possible	$(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30})$

(α, β, γ) not all rational multiples of π

Analyzed by Miklos Laczkovich 1995:

ABC	the tile
equilateral	$\alpha = \pi/3$
equilateral	$\gamma = 2\pi/3$
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
$(\alpha, \alpha, \alpha + \beta)$	$\gamma = 2\alpha$
several possible shapes	$3\alpha + 2\beta = \pi$
several possible shapes	$\gamma = 2\pi/3$

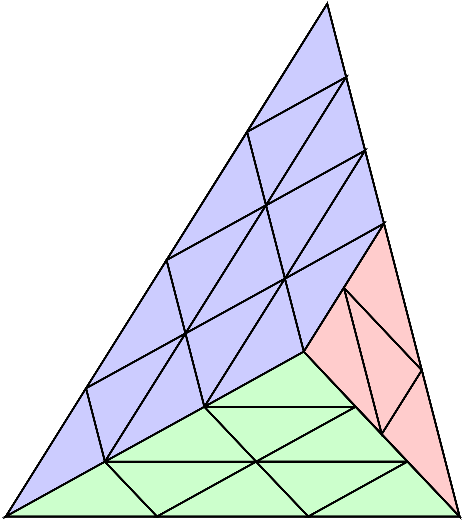
The case $3\alpha + 2\beta = \pi$

In Fall 2011, I took up the case $3\alpha + 2\beta = \pi$. The first shape of ABC that I considered was $(2\alpha, \beta, \alpha + \beta)$.

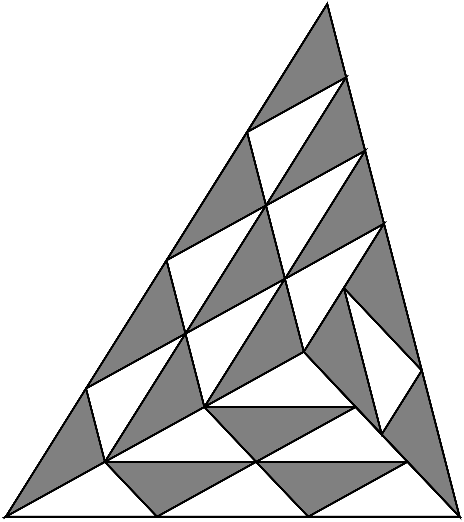
- I tried hard to prove no such tilings existed.
- I found some necessary conditions so I knew that N had to be at least 28.
- I also found equations that told me what shape the tile had to be, namely $(2, 3, 4)$.
- Eventually I wrote a computer program which was supposed to show there was no 28-tiling.
- On October 11, 2011, with the help of that program, I found that I had been wrong: there do exist such tilings.

This was the first of many surprises in this subject.

A 28-tiling



Some (not all) tilings can be colored black and white



The coloring equation

Let M be the number of black tiles minus the number of white tiles. Let (X, Y, Z) be the lengths of the sides of ABC, with Y opposite B. Suppose there is only one tile at B and it is black. Then

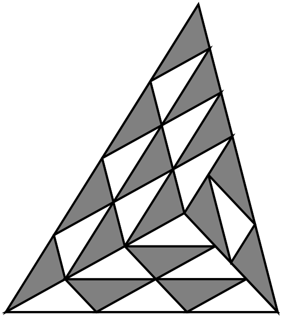
$$M(a + b + c) = X \pm Y + Z$$

Coloring equation example

$$M = 2 = 15 - 13$$

$$M(a + b + c) = X - Y + Z$$

$$M(2 + 3 + 4) = 18 = 16 - 12 + 14$$



The area equation

N times the area of the tile = area of ABC.

Twice the area of a triangle is the product of two sides times the sine of the included angle. Hence, the area equation:

$$XZ \sin B = Nac \sin \beta$$

If, for example, the angle at B is β then

$$XZ = Nac$$

Rationality of the tile

- "The tile is rational" means a/c and b/c are rational. Then, after scaling, we can assume (a, b, c) are integers with no common divisor.
- With the area equation and the coloring equation, you could get started using some number theory if you only knew that the tile is rational.
- You will need to assume the angles are not rational multiples of π if you hope to prove the tile is rational.
- It is difficult to prove that the tile has to be rational. Laczkovich 2012 did it for certain of the possible shape templates, but not all. I proved it for some other cases, including all the $3\alpha + 2\beta$ cases. It is still unknown for at least one other case.

Tiling equations

For each possible shape of ABC, we can combine the coloring equation, the area equation, and the law of sines to express (X, Y) in terms of the angles of ABC, and get a "tiling equation" that is a necessary condition for a tiling.
For example if ABC has angles $(2\alpha, \beta, \alpha + \beta)$ then the tiling equation turns out to be, for some K we have

$$N + M^2 = 2K^2$$

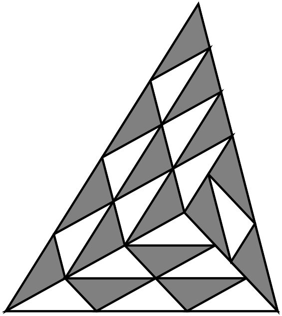
Here M is the coloring number, and we can show $M^2 < N$. The number K is determined by $s = a/c = K/M$. The law of cosines tells us $b/c = 1 - s^2$, so that determines the tile, up to scaling.

The tiling equation for $N = 28$

$$N + M^2 = 2K^2$$

$$28 + 2^2 = 2 \cdot 4^2$$

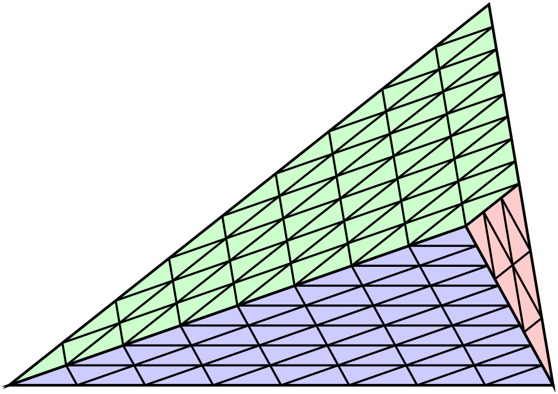
So $M = 2$ and $K = 4$.
$(a, b, c) = (2, 3, 4)$ so $s = a/c = M/K$.



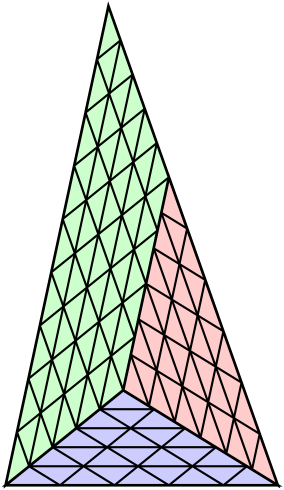
The triquadratic tilings

The 28-tiling is just the first member of a family of *triquadratic tilings*. These are called that because they are almost composed of three quadratic tilings, but one of the three is missing a few tiles.

A triquadratic tiling with $N = 153 = 9 \cdot 17, M = 3, K = 9$



A triquadratic tiling with $N = 126 = 9 \cdot 14, M = 6, K = 9$



In most cases, N and ABC determine the possible tile(s)

For example, in the case we discussed, given N, we have bounds on the possible K and M, so we can search for the possible solutions of $N + M^2 = 2K^2$.

- If there are no solutions, there is no corresponding N-tiling
- If there are solutions, there may or may not still be N-tilings.
- But there will be, for given N, only finitely many (usually just one) shapes of tile and ABC to consider.
- So in principle, we could check the remaining cases by trial-and-error. It is just a jigsaw puzzle, with N identical pieces. Can you fit them together in the frame ABC?

That is how the 28-tiling and some other tilings were discovered.

Other tiling equations

Define $s = a/c$. Then given N, and a pattern for the shape of ABC, such as $(2\alpha, \beta, \alpha + \beta)$, or "equilateral", or "isosceles with base angles α", we do the following:

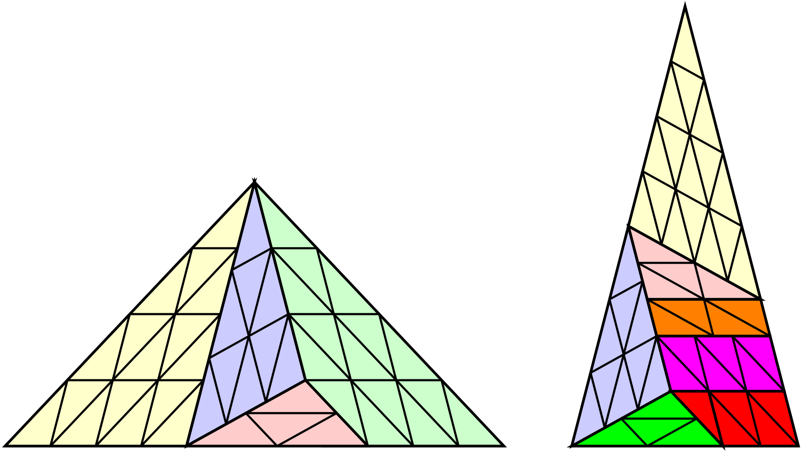
- Derive a tiling equation for that shape.
- Make a table of its solutions, showing the possible tiles for each N, and ruling out many N for which there is no solution.
- Try to construct tilings when there is a solution.
- Try to prove there is no tiling when you can't construct one.

By now, I have a hundred pages of such tables, tilings, and non-existence proofs, but there are still a lot of question marks in those tables.

Possible N-tilings with $3\alpha + 2\beta = \pi$ for $N \leq 84$

N	M	(a, b, c)	(A, B, C)	tiling exists
26	4	(3, 8, 9)	isosceles-β	no
28	2	(2, 3, 4)	triquadratic	yes
39	7	(12, 7, 16)	isosceles-β	no
44	6	(2, 3, 4)	isosceles-β	yes
45	3	(6, 5, 9)	isosceles-$\alpha + \beta$	?
47	5	(4, 15, 16)	isosceles-β	no
48	4	(2, 3, 4)	isosceles-$\alpha + \beta$	yes
59	9	(20, 9, 25)	isosceles-β	?
66	8	(15, 16, 25)	isosceles-β	?
70	8	(6, 5, 9)	isosceles-α	?
71	7	(10, 21, 25)	isosceles-β	?
72	6	(3, 8, 9)	isosceles-$\alpha + \beta$	?
74	6	(5, 24, 25)	isosceles-β	?
75	5	(2, 3, 4)	isosceles-$\alpha + \beta$	?
77	5	(2, 3, 4)	$(2\alpha, \alpha, 2\beta)$	?
83	11	(30, 11, 36)	isosceles-β	?
84	10	(2, 3, 4)	isosceles-α	yes

Tilings with $N = 44$ and 48, with $3\alpha + \beta = \pi$ and tile $(2, 3, 4)$



No 7-tiling

- In December, 2018, I found a short, computationally-abetted proof that there is no 7-tiling or 11-tiling.
- It does not use the aforementioned hundred pages of proofs and calculations.
- It does use the previous work of Laczkovich and Snover.
- I will show you how this proof goes.

The first cases to consider

- ABC similar to the tile.
  Done: N is a square or a sum of two squares, or three times a square [Snover 1991].
  Hence N is not a prime congruent to 3 mod 4.
- (α, β, γ) all rational multiples of π.
  Laczkovich reduced the possibilities to two cases:
- ABC is isosceles and the tile is a right triangle. Then N is $2k^2$, $6k^2$, or $2(e^2 + f^2)$. Thus even, and not a prime.
- ABC is equilateral and the tile is isosceles with base angles $\pi/6$. Then N is $3m^2$, which can't be prime.

When not all angles are rational multiples of π

ABC	the tile
equilateral	$\alpha = \pi/3$
equilateral	$\gamma = 2\pi/3$
isosceles, base angles β	$(\alpha, \beta, \pi/2)$
isosceles, base angles α	$\gamma = 2\alpha$
$(2\alpha, \beta, \alpha + \beta)$	$3\alpha + 2\beta = \pi$
$(2\alpha, \alpha, 2\beta)$	$3\alpha + 2\beta = \pi$
isosceles	$3\alpha + 2\beta = \pi$
five more shapes	$\gamma = 2\pi/3$

Tile not rational, ABC isosceles $(\beta, \beta, 2\alpha)$, $\gamma = \pi/2$

- Then N is $2k^2$ or twice a sum of two squares. 2018
- In particular $N > 3$ is not prime, since it is even.
- If $N/2$ is not a square, the tile is rational;
- $\tan \beta = e/f$ where $N = e^2 + f^2$, as you see in the biquadratic tilings.

The case $3\alpha + 2\beta = \pi$

- There are several possible shapes of ABC.
- For each one, I have a tiling equation that rules out $N = 7$ and $N = 11$.
- But in order not to depend on a long unpublished work, I don't use those results.
- Instead, a direct computational proof, aided by a lemma.
- The lemma: there must be at least two c edges on each side of triangle ABC, when angle B of ABC is α or β.

The idea of the computation

In short: Compute all possible $\mathbf{d}$-matrices and check the area equation and coloring equation. If none works there is no tiling.

$$\begin{aligned} X &= pa + qb + rc \\ Z &= ua + vb + wc \\ Y &= ka + \ell b + mc \end{aligned}$$

Substitute these expressions for (X, Y, Z) in the coloring equation. With $P = p + u \pm k$, $Q = q + v \pm \ell$, $R = r + w \pm m$ we have

$$M(a + b + c) = X \pm Y + Z = Pa + Qb + Rc.$$

Dividing by c and use $a/c = s$ and $b/c = 1 - s^2$ we have

$$M(2 + s - s^2) \quad = \quad Ps + Q(1 - s^2) + R$$

The idea of the computation, continued

With $s = a/c$ we have

$$M(2 + s - s^2) \quad = \quad Ps + Q(1 - s^2) + R$$

For given (M, P, Q, R) that quadratic can be solved for s (provided its discriminant is nonnegative). The area equation too can be expressed in terms of s, and we can check if it is satisfied for the s from the coloring equation. For a given N, we need to consider only values of the integer parameters between 0 and N. The point is this: if there is an N-tiling, these equations have a rational solution (since the tile has to be rational). So, if they have no rational solutions, there is no N-tiling.

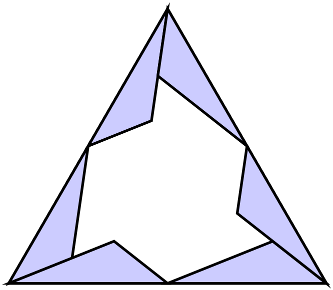
SageMath

- SageMath is a public-domain computer algebra and number theory system. It packages together a number of special-purpose programs. You can download and install it and learn to use it. Maybe it will help you.
- It knows about algebraic number fields. You can ask it if a certain number is rational or not.
- I programmed the computation described on the previous slide in a few minutes. It ran for $N = 7$ in 27 seconds and three minutes for $N = 11$.
- No solutions were found. Therefore there are no 7-tilings or 11-tilings with $3\alpha + 2\beta = \pi$.
- Technically it's two different programs, when ABC is isosceles or not.
- It doesn't work for $N = 19$ as some solutions are found. They don't correspond to tilings, but that isn't proved by the computation.

When the tile has an angle $\gamma = 2\pi/3 = 120^\circ$

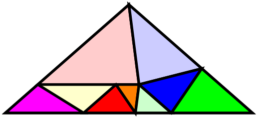
- The other two angles can be arbitrary (but not rational multiples of π).
- There are five possible shapes of ABC to consider.
- A boundary vertex can be composed of angles contributed by 3 or 6 tiles.
- Hence the tiles can't necessarily be colored black and white usefully. There is no coloring equation.
- We can still prove N can't be 7 or 11 by computation.
- The idea is that each side of ABC is at least $2c$ in length, and that makes the area be more than 12 tiles.

The idea for the case $\gamma = 2\pi/3$



- Six tiles have to be there because each side is at least $2c$.
- Now the area left to tile is more than six tiles.
- The computation can be done by hand in two pages, and checked in SageMath.

The case $\gamma = 2\alpha$ and ABC isosceles and α/π irrational



- So $\pi = \alpha + \beta + \gamma = 3\alpha + \beta$.
- So the vertex angle of ABC is $\alpha + \beta$.
- Laczkovich proved there are lots of such tilings.
- But the smallest one I can find has more than five million tiles.
- I can prove $N \geq 12$, just by geometry–no calculations needed.
- It's a jigsaw puzzle argument.
- Good thing because there's no coloring equation.
- That's a big gap! from 12 to five million.
- I don't know if there is a 19-tiling in this case or not.

Tiling an equilateral ABC

- If ABC is equilateral, there are two possibilities for the tile: one angle is $\pi/3$, or one angle is $2\pi/3$.
- The other two angles can be arbitrary (but not rational multiples of π).
- Every such tile can be used to tile *some* equilateral ABC, but N may be quite large.
- The smallest one I have constructed has $N = 10395$.
- It has one angle $2\pi/3$. I already showed you the picture.
- With one angle $\pi/3$, the smallest one I can construct has more than 32,000 tiles. It's too big to draw!
- But the task at hand is to prove N can't be 7 or 11.
- We did that already for $\gamma = 2\pi/3$.
- I did the $\pi/3$ case by another computation.

Digression on equilateral tilings–for the number theorists

- In December, 2018, I proved that if an equilateral triangle is N-tiled and $N > 3$, then N is not prime.
- Perhaps that could be generalized to N of the form, pk^2, where p is prime.
- That problem is connected to elliptic curves. Equilateral ABC can be Nk^2 tiled, for some k, if and only if one of these elliptic curves has a finite rational point [Laczkovich, private communication]

$$y^2 = Nx(x-1)(x-4)$$
$$y^2 = -Nx(x-1)(x-4)$$

- A lot is known about elliptic curves (UCSC has some experts!) and SageMath has a lot of built-in knowledge, including the "Cremona database".

Putting it all together

Theorem

There is no 7-tiling or 11-tiling of any triangle by any tile.

Proof. Laczkovich reduced the problem to a finite number of cases. I dealt with each possible case. QED. Here's the list of cases again:

ABC	the tile	proof method
equilateral	$\alpha = \pi/3$	computation
equilateral	$\gamma = 2\pi/3$	area requires ≥ 12 tiles
five more shapes	$\gamma = 2\pi/3$	same area computation
isosceles, base angles β	$(\alpha, \beta, \pi/2)$	proved N is even
isosceles, base angles α	$\gamma = 2\alpha$	jigsaw puzzle proof
five shapes	$3\alpha + 2\beta = \pi$	computed all $\mathbf{d}$-matrices
		and checked area
		and coloring equations