Triangle Tiling

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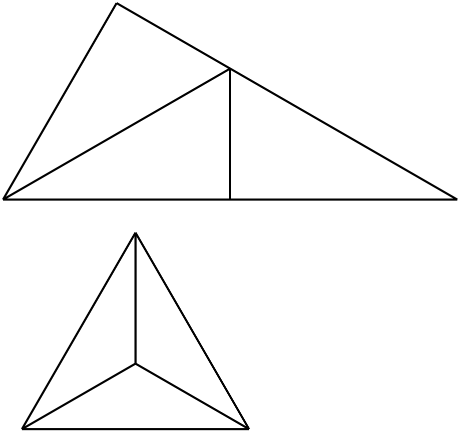
Definition of N-tiling

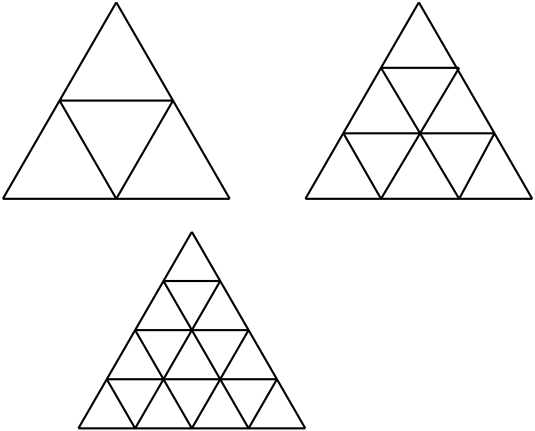
An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile".

The tile may or may not be similar to ABC. We wish to understand possible tilings by completely characterizing the triples (ABC, T, N) such that ABC can be N-tiled by T. In particular this understanding should enable us to specify for which N there exists a tile T and a triangle ABC that is N-tiled by T; or given N, determine which tiles and triangles can be used for N-tilings; or given ABC, to determine which tiles and N can be used to N-tile ABC.

Two 3-tilings

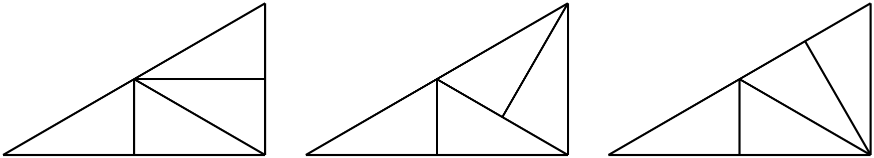


A 4-tiling, a 9-tiling, and a 16-tiling

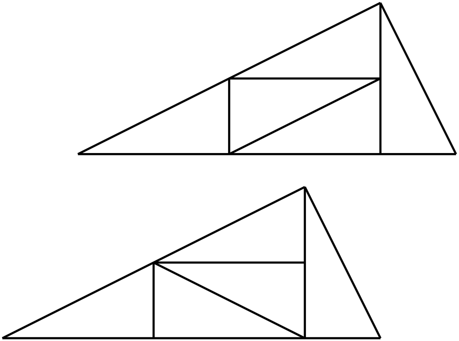


This illustrates the *quadratic tilings*, possible when N is a square, for any triangle.

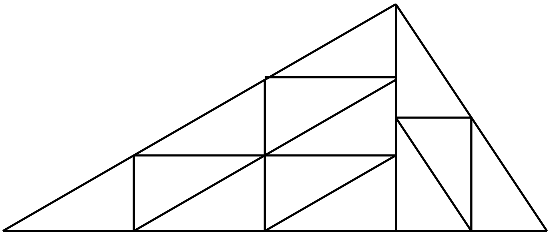
Three 4-tilings



Two 5-tilings

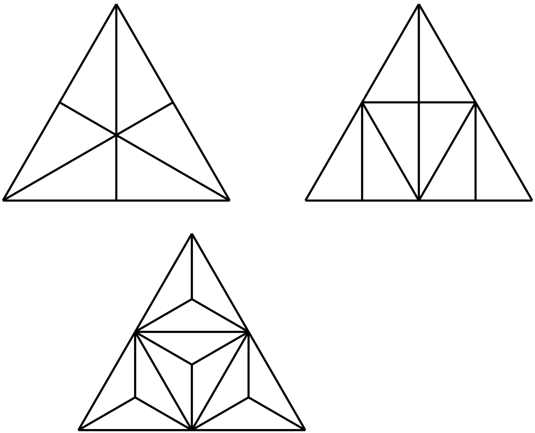


A 13-tiling

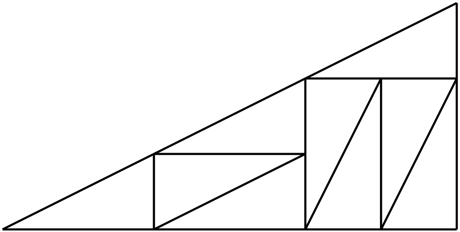


This illustrates the *biquadratic tilings*, possible when $N = e^2 + f^2$ using a right triangle of sides e and f.

A 6-tiling, an 8-tiling, and a 12-tiling

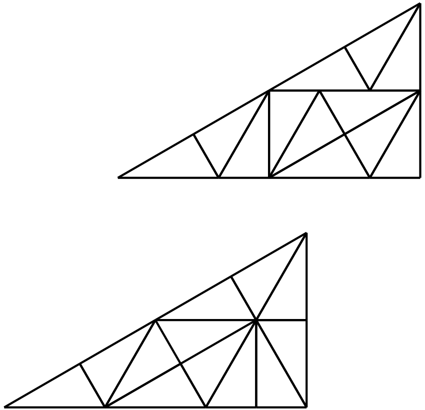


Another 9-tiling



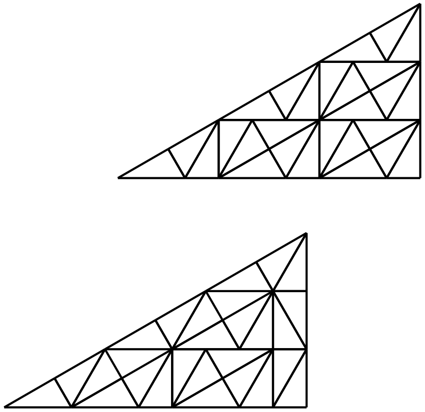
The $3m^2$ tilings

$N = 3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. Here is the case $m = 2$ and $N = 12$:



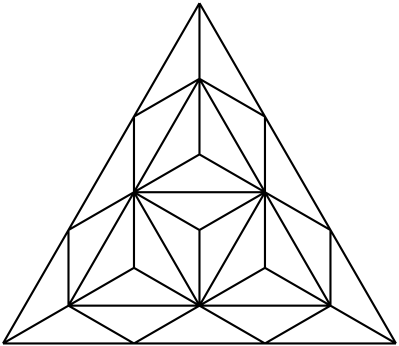
The $3m^2$ tilings

Here is the case $m = 3$, $N = 27$:



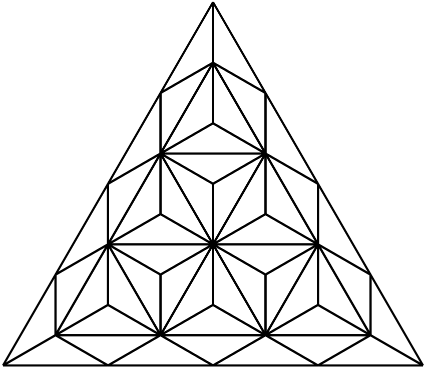
Until October 12, 2008, no examples were known of more complicated tilings than those illustrated above.

A prime 27-tiling



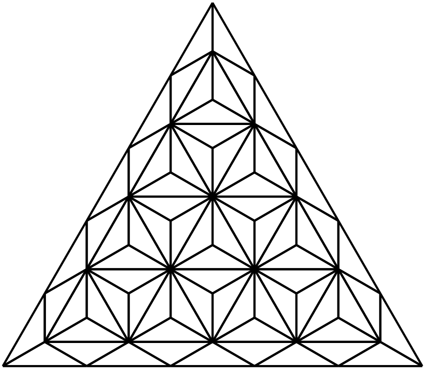
Then we found a family of $3m^2$ tilings, built from hexagons.

$3m^2$ tilings for $m = 4$, $N = 48$



This tiling is made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides.

$3m^2$ tiling for $m = 5$, $N = 75$



In general one can arrange $1 + 2 + \ldots + k$ hexagons in bowling-pin fashion, and add $k + 1$ tiles on each of three sides, for a total number of tiles of

$$6(1 + 2 + \ldots + k) + 3(k + 1) = 3k(k + 1) + 3(k + 1) = 3(k + 1)^2.$$

No 7-tiling

There is no 7-tiling. There is also no 11-tiling, no 14-tiling, no 17-tiling, no 19-tiling, no 23-tiling. Is there a 28-tiling? At the time I posted the abstract for this talk I did not know, but you will find out today. None of these theorems is trivial. All of them are special cases of much more general theorems.

Two $25 Erdős Problems

- Find all positive integers N such that at least one triangle can be cut into N triangles congruent to each other.
- Find (and classify) all triangles that can only be cut into n^2 congruent triangles for any integer n.

We have made progress on both of these problems (mentioned in Soifer's book, *How does one cut a triangle?*, p. 48).

We proved that, with one possible exception, the families of tilings we have exhibited today are all the possible triangle tilings. More specifically, if there is an N-tiling of ABC by a tile T, then there is an N-tiling of ABC by T that belongs to one of the exhibited families, unless T has a 120 degree angle and (up to scaling) integer side lengths, and in that case, $N \geq 96$.

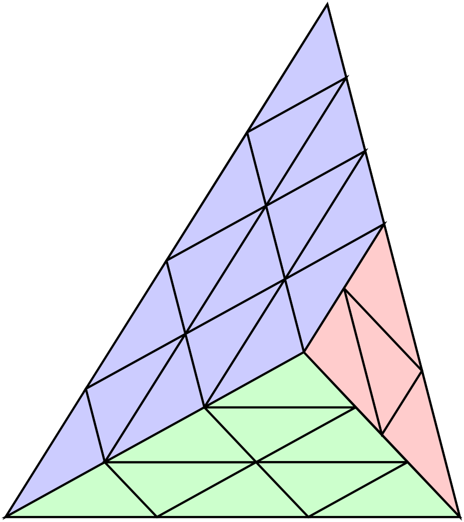
Proof techniques

- Linear algebra and eigenvalues for the case T similar to ABC.
- Field theory and the algebraic number theory of cyclotomic fields, when the sides of the tile are not commensurable.
- Counting arguments based on the way angles of tiles occur at vertices of the tiling.

The case $3\alpha + 2\beta = \pi$

Those techniques enabled us (in about 120 pages) to *almost* prove the theorems we wanted. However, one case resisted. Namely, when $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π. (The tile has angles α, β, and γ, and sides a, b, and c.) In that case the triangle ABC has to have an angle 2α at A, $\alpha + \beta$ at C, and β at B. We tried hard to prove no such tilings existed. Eventually we wrote a computer program which was supposed to show there was no 28-tiling. On October 11, 2011, with the help of that program:

A 28-tiling



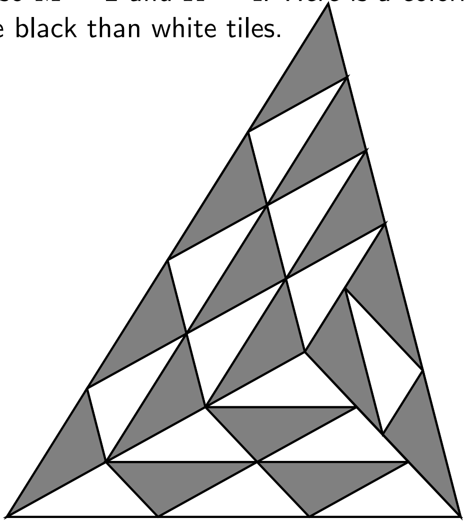
The tiling equation

$$N + M^2 = 2K^2 \qquad \text{and } M^2 < N$$

Here M is the excess of the number of black tiles over white in a correct coloring with the tile at B black. We can take $a = M$ and $c = K$. Then $b = N/K - K = K - M^2/K$.
Hence b is an integer if and only if K divides M^2.

The tiling equation for $N = 28$

$28 + 2^2 = 2 \cdot 4^2$, so $M = 2$ and $K = 4$. Here is a coloring of the tiling with 2 more black than white tiles.



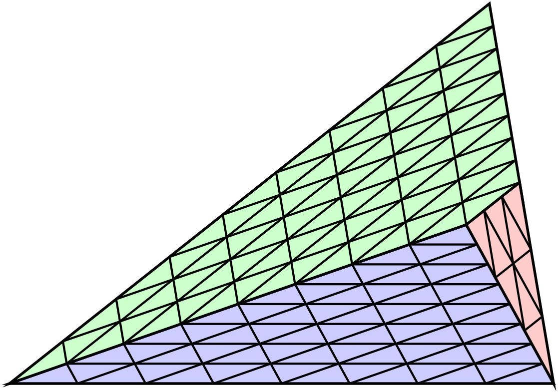
The triquadratic tilings

If K divides M^2, and $N + M^2 = 2K^2$, then there is an N-tiling of ABC by the tile with $a = M$ and $c = K$ and $b = K - M^2/K$. These are called *triquadratic tilings* because they are almost composed of three quadratic tilings, but one of the three is not quite all there.

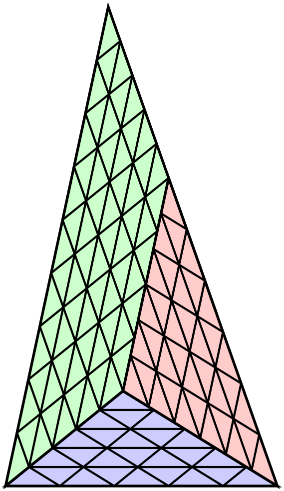
M and K determine the tile

For each N there are finitely many possible solutions of $N + M^2 = 2K^2$ with $M^2 < N$. The least N such that the equation has more than one solution in which K divides N (or equivalently, M^2) is $N = 87808$. In that case there are two solutions with K dividing N, namely $M = 112, K = 224$ and $M = 208, K = 256$. These correspond to the tiles $(112, 119, 224)$ and $(208, 87, 256)$ respectively.

A triquadratic tiling with $N = 153 = 9 \cdot 17, M = 3, K = 9$



A triquadratic tiling with $N = 126 = 9 \cdot 14, M = 6, K = 9$



For which N is there an N-tiling of some ABC by some tile?

- N is a square or a sum of two squares
- N has the form $3m^2$ or $6m^2$ for some integer m and ABC is equilateral; and we can say what the tile has to be.
- N is twice a square, or six times a square, or twice a sum of two squares; ABC is isosceles, having angles α, α, and 2β, or angles β, β, and 2α, and the tile is a right triangle similar to half of ABC.
- $3\alpha + 2\beta = \pi$, triangle ABC has angles 2α, β, and $\beta + \alpha$, and $\sin(\alpha/2)$ is rational. N must be a square times a product of distinct primes which are 2 or of the form $8n \pm 1$, and the tiling equation $N + M^2 = 2K^2$ has solutions with $K|M^2$.
- Possibly if T has a 120 degree angle and integer side lengths and $N \geq 96$, although no such tilings are known.
- *And in no other case!*