ASSIGNMENT 10: PEANO ARITHMETIC

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The axioms of Peano Arithmetic (**PA**) are listed on page 82 of Kleene.

1. Explain the justification given for line 7 of the example proof on page 84. Here "explain" means to indicate the matching of specific formulas in line 7 to the variables written with capital letters on page 82.

2. Exhibit a proof in **PA** of the formula $2 + 2 = 4$. Here 2 is $0''$ and 4 is $0''''$.

3. Prove the theorem given in lecture, that if $k = Val(t)$ for a closed term t, then $\vdash t = \bar{k}$. *Hint*: Use induction on the complexity of the term t. You will see that the axioms of **PA** are very convenient for this proof.

4. The "induction rule" says that if $A(0)$ and $A(x) \supset A(x')$ are provable, then $A(x)$ is provable. Show that the induction rule is valid for **PA**. (Hint: this is sketched on p. 181, but I want to see more detail.)

5. Let $\mathbf{PA}^*$ be defined by removing the induction axiom (Axiom 13, p. 82) and replacing it with the induction rule (problem 3, or p. 181). Show that the induction axiom is provable in $\mathbf{PA}^*$, so the two systems actually have the same theorems.

6. Course-of-values induction. See paragraph 2, page 193 of the textbook. Show that every instance of course-of-values induction is a theorem of **PA**. The proof is sketched in two lines on p. 193, so all you have to do is flesh that sketch out, providing a few more details.

7. Prove that a predicate is representable if and only if its representing function is representable.

8. Show that $\mathbf{PA} \vdash \forall x\,(x \neq 0 \supset \exists u\,(u' = x)$. *Hint*: The proof will have to use an instance of the mathematical induction schema in **PA**. Observe the difference between using induction formally (within **PA**) and using it informally.

9. Recall that $x < y$ is an abbreviation for $\exists z(z' + x = y)$. Show that **PA** proves this is equivalent to $\exists z(x + z' = y)$.

10. Using the result of problem 9, show that for each numeral $\bar{n}$,

$$\mathbf{PA} \vdash x < \bar{n}'' \equiv x < \bar{n}' \vee x = \bar{n}'.$$

Hint: Replace the formulas involving $<$ by their true meanings involving $\exists$. Then prove both directions of the equivalence (informally, it is not required to write out an official formal proof). Then say "this proof can be done in **PA**" and wave your hands.

11. Show that for each numeral $\bar{n}$,

$$\mathbf{PA} \vdash x < \bar{n} \equiv x = 0 \vee x = \bar{1} \vee \ldots \vee x = \bar{n}.$$

Hint: Proceed by informal induction on n. Use problem 10 for the induction step. Notice the difference between using induction formally (within **PA**) and informally, as here.

12. Show that the theorem in problem 11 can be proved in **PA** *without* induction (for each numeral $\bar{n}$) using Kleene's definition of $x < y$, if we assume the result of problem 8 (every nonzero integer is a successor). In other words, we need induction *only* for that result. *Hint*: Fixing n, argue in **PA** as follows. If $x = 0$, we're done. So x is a successor, $x = y'$. If $y = 0$ then $x = \bar{1}$ and we're done. And so on, for n steps, each step decreasing the numeral on the right of the equation, until it comes to zero. You can find a rather formal version of this on p. 198 of Kleene.