ASSIGNMENT 11: PEANO ARITHMETIC AND THE PRIMITIVE RECURSIVE FUNCTIONS

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1. The function $\exp(a, b) = a^b$ is primitive recursive, and according to our general theory, it is representable. This exercise asks you to explicitly work out the formula $A(x, y, z)$ that represents the exp function. You may use the formula $B(c, d, i, u)$ that represents Gödel's β function in your answer.

2. A function f from $\mathbb{N}$ to $\mathbb{N}$ is called "provably total in **PA**" if there is a formula $A(x, y)$ such that $A(\bar{x}, \bar{y})$ holds in $\mathbb{N}$ if and only if $y = f(x)$, and $\mathbf{PA} \vdash \forall x \exists! y\, A(x, y)$.

(i) Write down for contrast the definition of "representable" and compare it with the definition given above of "provably total". Explain the difference in English.

(ii) Show that if f is provably total, then f is representable.

(iii) Observe that there is no obvious reason why representable should imply provably total. Explain briefly.

3. Show that not every function f from $\mathbb{N}$ to $\mathbb{N}$ is provably total in **PA**. (To avoid any confusion: "function from $\mathbb{N}$ to $\mathbb{N}$" implies that $f(n)$ is defined and is a natural number for each natural number n, so we are asking you to show that not every total number-theoretic function is *provably* total.)

4. Show that every primitive recursive function is provably total in **PA**. To avoid some possible confusions:

(i) Note carefully that this is *not the same* as showing each primitive recursive function is representable. For example, if we delete the induction schema from **PA** and add the axiom $x \neq 0 \supset \exists y (y' = x)$, we get a theory in which every primitive recursive function is representable, but most of them are not provably total.

(ii) Note also that the problem does not just ask you to give an informal proof that each primitive recursive function is total. You have to show that each primitive recursive function is *provably* total.

5. Is Ackermann's function provably total in **PA**? Justify, or at least explain, your answer.

6. Show that Ackermann's function is representable in **PA**.

7. Consider the "Collatz function", $f(x)$, defined as follows. Given x, let $x_0 = x$, and

$$x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n \text{ is even} \\ 3x_n & \text{if } x_n \text{ is odd} \end{cases}$$

Then $f(x)$ defined as the least n such that $x_n = 1$, if there is such an n, and otherwise $f(x)$ is undefined. It is presently unknown whether f is a total function or not, though it has been verified numerically that if there is some x for which $f(x)$ is undefined, that x is quite large!

Show that if f is total, it is representable in **PA**. This is true even if we can't settle the question whether f is or is not total. The point of this exercise is to emphasize the difference between "provably total" and "representable."

8. (Gödel's β function). Specialize to the case of sequences of length 3.

(i) Write out the reduction of $\exists a, b, c$ to

$$\exists c, d\, (\beta(c,d,0) = a_0 \wedge \beta(c,d,1) = a_1 \wedge \beta(c,d,2) = a_2.$$

(ii) Take the sequence of a_i to be $2, 5, 3$. What c and d will be produced by the proof in Kleene § 48? Compute c and d, then compute $m_i = \delta(d,i)$ for $i = 0, 1, 2$ and verify that m_0, m_1, and m_2 are relatively prime, and verify that $c \mod m_0 = 2$, $c \mod m_1 = 5$, and $c \mod m_2 = 3$.

9. Do we, or do we not, need to check that the Chinese remainder theorem is provable in **PA** to prove that every primitive recursive function is representable in **PA**? Justify or explain your answer. *Hint*: There is some related discussion at the very end of § 49, but that discussion may possibly only confuse you, so I advise you not to look there until you have figured out the answer. Instead, ask yourself where the main argument for the theorem that every primitive recursive function is representable is taking place. Is it an informal argument, or an argument within **PA**?