ASSIGNMENT 13: THE FIRST INCOMPLETENESS THEOREM

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1. Show that if $x^3 + y^3 + z^3 = 30$ has any solutions in positive or negative integers (x, y, z), then that fact can be expressed and proved in **PA**.

2. Is it possible, as far as you know (and I am assuming you are not an expert in number theory) that the statement that the equation in problem 2 has no solutions is unprovable in **PA**? (Assume that we don't know of any solutions to that equation or any theorems that would show there are none.)

3. Introduce ϕ by the self-reference lemma to say "I am provable" (instead of "I am not provable" as for the incompleteness theorem).

(a) Write out explicitly the "equation" that ϕ and its Gödel number have to satisfy.

(b) Can you determine in a few minutes whether ϕ is true or false, just on the basis that it satisfies the "equation"? Why or why not? Comment further if you wish.

4. (Mutual self-reference). Let P and Q be two formulas with two free variable x, y. Show that there exist two sentences ϕ and ψ such that ϕ says, "We have the property P" and ψ says, "We have the property Q", in the sense that

$$\vdash \phi \equiv P(\overline{\ulcorner \phi \urcorner}, \overline{\urcorner \psi \urcorner})$$

and

$$\vdash \psi \equiv Q(\overline{\ulcorner \phi \urcorner}, \overline{\urcorner \psi \urcorner})$$

Hint: Imitate the proof of the self-reference lemma. You may use the notation $C_n(\bar{n}, y)$ for the formula with Gödel number n of two free variables x and y, with $\bar{n}$ substituted for the free variable x; that will be a formula of two free variables n and y as shown much more precisely in the lectures. With this notation the proof of the self-reference lemma itself looks like this (with C_n now enumerating formulas of one free variable instead of two):

$$\begin{aligned}
\chi(n) &= A[x := \overline{\ulcorner C_n(n) \urcorner}] \quad \text{by definition} \\
\phi &= \chi[n := \overline{\ulcorner \chi \urcorner}] \quad \text{by definition} \\
&= A[x := \overline{\ulcorner C_{\ulcorner \chi \urcorner}(\overline{\ulcorner \chi \urcorner}) \urcorner}] \\
&= A[x := \overline{\ulcorner \chi(\overline{\ulcorner \chi \urcorner}) \urcorner}] \\
&= A[x := \overline{\ulcorner \phi \urcorner}]
\end{aligned}$$

Imitate this proof to solve the exercise.

5. Show that there are two true sentences ϕ and ψ of **PA** such that neither one implies the other in **PA** (and hence of course each is unprovable). *Hint*: Use the previous exercise.

6. Use the result of problem 5, together with the completeness theorem and Löwenheim-Skolem theorems, to show that two non-standard models of **PA** need not be isomorphic as models of **PA**. By definition, a non-standard model is one that is not isomorphic to the standard model.

7. (Logical form and independence). The concept of Σ^0_1 formula (and hence sentence) was defined in lecture, where we proved that a true Σ^0_1 sentence is provable. A Π^0_1 formula is of the form $\forall x_1, \ldots, x_n A(x_1, \ldots, x_n)$.

(a) Show that a Π^0_1 formula is provably equivalent to the negation of a Σ^0_1 formula and vice-versa. (You did this already in an earlier assignment, so no need to repeat it.)

(b) Show that the sentence produced by Turing's proof of the incompleteness theorem is Π^0_1.

(c) Show that the sentence produced by Gödel's proof of the incompleteness theorem is Π^0_1.

(d) Show that an unprovable Π^0_1 sentence is necessarily true.