ASSIGNMENT 14: THE SELF-REFERENCE LEMMA

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1. (ω-consistency). (a) Show that an ω-consistent theory T is consistent.
(b) Show that consistency does not imply ω-consistency. *Hint*: Make use of the true unprovable formula from the first incompleteness theorem.

2. (Kleene's textbook proof of the incompleteness theorem.) Kleene outlines the proof of the First Incompleteness Theorem in §42, starting on page 204. He gives the proof fairly precisely modulo the as-yet-unproved Lemma 21, bottom of page 206. In this exercise you will compare that proof with the one given in lecture.

(a) Write out the formulas A and B required in Lemma 21, using the predicate Prf and the functions $Subst$ and Num introduced in lecture. Assume for this part of the exercise that we have extended **PA** by function symbols for $Subst$ and Num.

(b) Now introduce letters S and $Numgraph$ for formulas representing $Subst$ and Num, and write out the formulas A and B of Lemma 21 using those formulas.

(c) Finally, write out the true unprovable sentence $A_p(\mathbf{p})$ introduced on page 207, as explicitly as possible using S and $Numgraph$.

(d) Does this turn out to be the same sentence produced in lecture by the self-reference lemma?

3. (Theorems with only long proofs). Refer to the lecture proof of Gödel's theorem that there are theorems whose only proofs are long. In that proof, such theorems are explicitly constructed using the self-reference lemma, and the argument shows that those theorems are true. In one lecture slide, therefore, I have proved these theorems are true, even though they supposedly have only very long proofs. Explain why this is not a contradiction. Specifically, what principles are used in that short proof on the slide, that cannot be formalized in **PA**?

4. In lecture we defined $x < y$ as an abbreviation for $\exists m\,(m' + x = y)$. Show that $\mathbf{PA} \vdash a < b \wedge b < c \supset a < c$. It is not requested that you write out a formal proof. Instead, write out an informal proof (probably 5 to 15 steps) that *could* be formalized.

5. Models of **PA**. A model of **PA**, of course, is a set M, with a distinguished element 0, and functions defined on M to interpret successor, addition, and multiplication. Let $\mathcal{M} = \langle M, O, plus, times, succ\rangle$ be such a model.

Show that there is a one-one homomorphism from the standard model into $\mathcal{M}$ ("isomorphism into" you might want to call it). Moreover this homomorphism is unique. $\mathcal{M}$ is called "non-standard" if this homomorphism is not onto.

6. Let $\mathcal{M}$ be a non-standard model of **PA**. Any element not in the range of the isomorphism from the standard model to $\mathcal{M}$ is called "non-standard." Elements in the range are "standard." Show that the standard elements are exactly the denotations of the numerals $\bar{m}$.

7. In problem 4, a formula was exhibited that defines $x < y$ in **PA**. Show that this formula always defines a linear order in every model $\mathcal{M}$ of **PA**.

8. Let $\mathcal{M}$ be a non-standard model. Show that every nonstandard element is greater than every standard element.

9. Show that there is no least nonstandard element.