ASSIGNMENT 15: THE SECOND INCOMPLETENESS THEOREM

MICHAEL BEESON

1. The *reflection principle* for **PA** is the schema (collection of sentences of the form)

$$\mathrm{Pr}(\overline{\ulcorner A \urcorner}) \supset A$$

Find an instance of the reflection principle for **PA** that is not provable in **PA**.

Hint: This is a two-or-three line problem. You do not need to go back to the self-reference lemma or make any complicated argument.

2. Let T be axiomatized by the axioms of **PA** plus the reflection principle for **PA**. Does Gödel's first incompleteness theorem apply to T?

3. (Partial truth definitions). The complexity of a formula is the number of its logical symbols (propositional connectives and quantifiers). Another way to define the complexity is inductively: atomic formulas have complexity zero, and the complexity of a compound formula is one plus the sum of the complexities of its constituents.

Prove that for each fixed positive integer n, there is a formula $\mathrm{Tr}_n(x)$ such that $\mathrm{Tr}_n(\bar{m})$ is true if and only if m is the Gödel number of a true sentence of complexity $\leq n$.

Hint: Proceed by induction on n. You only have to show how to define the formulas Tr_n. You do not have to show that $\vdash (Tr_n(\ulcorner A \urcorner) \supset A)$. This is also true, but it is harder than this homework exercise, because when you proceed by induction on the complexity of A, you need an induction hypothesis that applies to formulas with free variables. If you want to try to solve this harder problem, go ahead, but the point of the hint is that you do not need to deal with free variables in order to construct the requested formulas Tr_n.

4. Let $\mathbf{PA}_n$ be **PA** with the induction axioms restricted to those of complexity at most n.

(a) Show that $\mathbf{PA}_n$ is finitely axiomatizable. (Careful: there are infinitely many variables in **PA**, hence infinitely many instances of propositional axioms of small complexity.)

(b) It follows from a theorem of Gentzen (which we do not have time to cover in this course) that if a theorem A of complexity at most n has a proof in $\mathbf{PA}_n$, then it has a proof each of whose steps is of complexity at most $n + 20$. (See below for an explanation why the 20 is there.) Assume this theorem, and use it to show that **PA** is not finitely axiomatizable. *Hint*: Show that **PA** proves the consistency of $\mathbf{PA}_n$, using Exercise 3 and induction on the length of proofs of complexity at most n.

Remarks. First, about the 20. There is nothing special about 20; we may need some small constant to account for the fact that Gentzen used some proof rules slightly different from those of **PA**; in his system no constant is necessary but we need to translate his proofs into **PA**-style proofs, which might make some steps a little longer; hence the 20.

The second remark: I do not know how to prove that **PA** is not finitely axiomatizable without Gentzen's result.