ASSIGNMENT 2: EXAMPLES OF FIRST-ORDER THEORIES

MICHAEL BEESON

A *strict linear order* is a structure $(A, <)$, where $x < y$ is a binary relation that is transitive and trichotomous. Transitive means that if $a < b$ and $b < c$ then $a < c$. Trichotomous means that exactly one of $a < b$, $a = b$, and $b < a$ holds.

1. Express the axioms for strict linear order in formal first-order logic. It is OK to use infix notation $a < b$ instead of $<(a, b)$.

There is also a theory formulated with $\leq$ instead of $<$, and technically perhaps we should keep repeating "strict linear order", but the use of the symbol $<$ makes it just an annoyance to do so. Henceforth in this assignment we just say "linear order."

2. A linear order may or may not have a "left endpoint" or a "right endpoint". Express each of these concepts by a formula of first-order logic.

3. Show that the theory of linear orderings is not complete. (Hint: consider endpoints.)

4. A linear order is called "dense" if for any two elements, there is a third element between the first two. Express density by a first-order formula.

5. Show that density is not a consequence of the axioms of linear order.

Consider two orders $(A, <)$ and $(B, <)$. (We use the same symbol for both orderings, hoping no confusion will arise.) A map ϕ from A onto B is an isomorphism if it is one-to-one and preserves order, i.e. $a < b \implies \phi(a) < \phi(B)$.

6. Show that not any two dense linear orderings without endpoints are isomorphic, by giving two examples of dense linear orderings that cannot be isomorphic.

7. Show that if any two countable dense linear orderings without endpoints are isomorphic, then the theory of dense linear orderings without endpoints is complete. (Hint, use the Löwenheim-Skolem theorem and the completeness theorem.) Note that problem 6 does not rule out the possibility that this theory might be complete.

From now on we write DLO to stand for "dense linear order without endpoints."

8. Let φ be an order-preserving one-to-one map ("partial isomorphism") from a finite subset of a DLO $(A, <)$ into a DLO $(B, <)$. Let c be an element of A not in the domain of φ. Show that φ can be extended to be defined on a, and still preserve order.

9. Suppose A and B are two countable DLOs. Show that they are isomorphic. Hint, use the "back-and-forth method" as follows: Enumerate A as $a_1, \ldots, a_n, \ldots$ and B similarly.

Then for each n, define partial isomorphisms ϕ_n from a subset of A to B and ψ_n from B to A such that ϕ_n and ψ_n are inverses. The trick is to define ψ_{n+1} in terms of ϕ_n and vice-versa, in such a way that every element of A and B eventually get included in the domains.

10. List all four complete extensions of the theory of dense linear orders. (Hint: consider endpoints.)