ASSIGNMENT 9: ELEMENTARY RECURSION THEORY

MICHAEL BEESON

In this assignment we follow tradition and use "recursive function" to mean Turing computable function, which have proved is the same thing as μ-recursive and everywhere defined. Similarly "partial recursive function" can be used for Turing computable partial function.

A set is called "recursively enumerable" (r.e.) if it is either empty or the range of a total recursive function. A set is called "semicomputable" if it is the domain of some partial recursive function. A set is recursive if its representing function is recursive.

1. Use the recursion theorem (instead of the diagonal method) to give a proof that the halting problem is recursively unsolvable. *Hint*: Suppose you had a solution of the halting problem (whether $\varphi_e(e)$ halts or not). Define f recursively so if it is ever defined, the result will be contradictory, but then use the hypothetical solution of the halting problem to ensure that it will be everywhere defined.

2. Call a set X of natural numbers $\Sigma^{\mathbf{0}}_{\mathbf{1}}$ if has a definition of the form

$$m \in X \leftrightarrow \exists \mathbf{x} A(m, x)$$

where $\mathbf{x} = x_1, \ldots, x_n$, and A is a bounded arithmetic formula. Call a set $\Pi^{\mathbf{0}}_{\mathbf{1}}$ if it has a definition in the form

$$m \in X \leftrightarrow \forall \mathbf{x} A(m, x).$$

(a) Prove that the complement of a $\Sigma^{\mathbf{0}}_{\mathbf{1}}$ set is $\Pi^{\mathbf{0}}_{\mathbf{1}}$ and vice-versa.

(b) Give an example of a $\Sigma^{\mathbf{0}}_{\mathbf{1}}$ set that is not $\Pi^{\mathbf{0}}_{\mathbf{1}}$. (Hint: Use the Kleene **T**-predicate, which we proved is definable by a bounded arithmetic formula.)

3. Prove that there exist lots of sets of natural numbers that are neither $\Sigma^{\mathbf{0}}_{\mathbf{1}}$ nor $\Pi^{\mathbf{0}}_{\mathbf{1}}$.

4. Prove that every nonempty semicomputable set is r.e. (as defined at the top of the assignment). *Hint*: Let X be the domain of φ_e. For $n = 1, 2, \ldots$, compute n steps in the computation of $\varphi(0), \ldots \varphi(n)$. As you get new answers, enumerate those elements. Use the **T**-predicate to make this argument precise.

5. Prove that every r.e. set is semicomputable.

6. Show that if a set X and its complement are both r.e., then X is recursive.

7. Suppose the set X is enumerated by an *increasing* recursive function. Show that X is recursive.

8. Show that every semicomputable set is $\mathbf{\Sigma^0_1}$. (Hint: Use the **T**-predicate.)