FINAL EXAM FOR PHIL 152

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Directions. This is an open-book, open-notes, open-homework. You can even search the Internet, but all the answers can be found in the lecture notes and homework or worked out in a couple of steps from things found there. You may not, of course, collaborate with other people. The exam is due on June 11 at noon. I do not seem to have a physical mailbox. Just push your exam under my office door, room 92E. Of course, scan-and-email is also fine. I will turn in the grades on June 12, and I will email you the results.

I think there is enough room on the exam to write the answers right on the exam, which makes them easier to grade, but if you would rather write on a separate page, that's OK. If you turn in a paper exam, please staple it together rather than dog-ear it.

1. (a) Prove that **PA** is consistent.

(b) Indicate which step or steps in your proof cannot be formalized in **PA**, and explain why not.

2. Who were the authors of *Principia Mathematica*? What was their aim in writing it?

3. What is "Skolem's paradox"? Why is it not really a paradox?

4. Prove that the predicate $n|m$ ("n divides m") is primitive recursive. You may appeal to any theorem proved in the lecture notes. (In the lecture notes, it is stated without proof that this predicate is primitive recursive.)

5. What is the "Church-Turing thesis"? Give two versions:

(a) An answer that does not mention human beings or the capabilities of their brains.

(b) An answer that does mention those things.

(c) Discuss the reasons for believing version (a), the reasons for believing version (b), and the possible reasons for doubting version (b).

6. Show that the Fibonacci numbers F_n can be computed by a Turing machine (from input n). Do not try to write a Turing machine to compute them, instead quote appropriate theorems. There are at least two possible solutions (other than writing a Turing machine). You may give more than one solution for extra credit.

7. What formula of **PA** represents the function $f(n) = 2^n$? You may use abbreviations such as A or B for formulas you need, but if you do you must define those formulas too, except for $x < y$, which you may use without writing it out explicitly.

8. Show that the set of (Gödel numbers of) theorems of **PA** is recursively enumerable (see Assignment 9 for the definition of "recursively enumerable.")

9. A number-theoretic function f is said to be *provably total in* **PA** if there is a Turing machine e that computes f (that is, $\varphi_e(n) = f(n)$ for all n) and $\mathbf{PA} \vdash \forall x \exists k \, \mathbb{T}(\bar{e}, x, k)$.

Define a function f that is Turing computable, but is not provably total in **PA**. (There are are least two different solutions; you may give more than one solution for extra credit.)

10. Assume that the definition of Turing machine has been modified as discussed in your homework so that Turing machines are "prefix-free", so that the concept "m is algorithmically random" makes sense using Turing machines for computation. Prove that the predicate "m is algorithmically random" is definable by a formula of **PA**. (You do not have to actually exhibit a defining formula, and you may assume that the **T**-predicate refers to the modified Turing machines.)

11. A *prime pair* is a pair of prime numbers, one of which is the other plus 2. (For example, 11 and 13.) The *twin-prime conjecture* states that there are infinitely many prime pairs. Exhibit a sentence of **PA** that expresses the twin-prime conjecture. Again, you may use abbreviations but the formulas abbreviated must be defined, except $x < y$.

12. A sentence ψ of **PA** is *independent of* **PA** if neither ψ nor $\neg\psi$ is a theorem of **PA**. At present no proof in **PA** (or elsewhere for that matter) is known of the twin-prime conjecture or its negation. Is it possible that we might be able to prove the twin-prime conjecture is independent of **PA** without settling the conjecture itself?