Computability and Logic, Phil. 152

- Computability, especially the limits thereof.
- Logic, especially, the limits thereof.
- Famous results on uncomputability and unprovability.
- Twice-weekly homework, please plan to work hard on it.
- Mathematical logic is not and cannot be a spectator sport.
- Final exam, but homework will count 80%
- Office hours 90-92E by appointment T-Th. 2 pm and 10 am are good times.

Textbook Details

- Textbook: Kleene's *Introduction to Metamathematics*.
- Ishi Press paperback edition has a foreword by Prof. Beeson.
- Foreword also available on Web, so any edition of Kleene is OK.
- Title is important as Kleene has another book that is not the right one.

Plan of the introduction

- Important ideas from the history of logic
- Some important people
- In your homework, you'll look many of these people up in Wikipedia, and identify the title and date of their most important publication.
- You'll also read the first part of Kleene.

Euclid: Father of the axiomatic method

- Euclid was Greek, but he lived and worked in Alexandria, Egypt, which was an important center of Greek intellectual life.
- The *Museum* was perhaps the first government-financed research institute.
- Euclid's *Elements* were a textbook summarizing the geometrical knowledge of the time.
- Euclid may have been the first to arrange the material in a deductive sequence
- Euclid starts from five "Postulates" and five "Common Notions", as well as a longer list of "Definitions."
- Every "Proposition" is (supposed to be) proved by logical reasoning from the postulates, common notions, and previous propositions.
- That is the "axiomatic method", which nowadays is employed in all of mathematics.

Euclid in education

- Until the early twentieth century, every educated person studied Euclid.
- Nowadays, some version of geometry is taught to the entire population, but not Euclid. The axiomatic method is no longer taught.
- You should make an acquaintance with Euclid part of your "liberal education."
- Consider buying the Green Lion Press edition, which is inexpensive and gives only Euclid, without the extensive commentary of other editions. This is just an informal, personal, recommendation: Euclid is only marginally and historically relevant to this course and will probably not be mentioned after today.

Euclid’s Definitions

Here are the first three of them:

- A **point** is that which has no part.
- A **line** is breadthless width.
- A **straight line** is a line which lies evenly with the points on itself.

Euclid realized that you have to start somewhere, and you should isolate the fundamental concepts you are going to reason about.

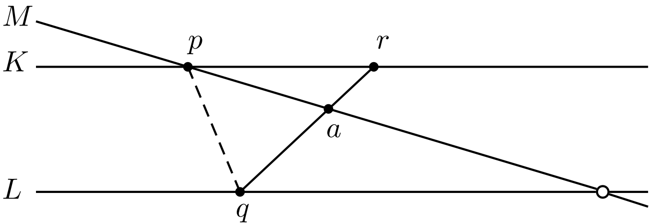
- The modern approach would be to take the fundamental concepts as *undefined*, and view these “definitions” as informal explanations of an “intended interpretation” of the language.
- This important shift of viewpoint took place only in the period 1870-1899.

Euclid's postulates

- To draw a straight line from any point to any point.
- To produce a finite straight line continuously in a straight line.
- To describe a circle with any centre and distance.
- All right angles are equal to one another.
- If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are angles less than two right angles.

Euclid's Postulate 5

Line pq falls on straight lines M and L making angles on the right side less than two right angles. The point indicated by the open circle is asserted to exist.



Line K can always be constructed parallel to L so the important part about Euclid 5 is that any *other* line through p has to meet L.

Doubts about Euclid 5

It was very early felt that Euclid 5 might not quite deserve the status of a "postulate."

- It seems less fundamental
- The meeting point might be very far away, so we can't just "see it in front of our faces".
- It would nicer to have a proof of Euclid 5 and not have to assume it.
- Or at least to replace it with something that seems more fundamental

Then millennia went by . . .

- The library at Alexandria was burned three times under dramatic circumstances (look it up!)
- Government support dried up, and the Empire itself fell
- Manuscripts made their way to India, including Euclid's *Elements*, but not including his book *Porisms* ("theorems"), which was irretrievably lost.
- The monsoon winds were known and in the spring a large fleet sailed from the Red Sea area for India, returning in the fall. This commerce carried Euclid, too.
- Euclid's *Elements* found its way to Arabia and Persia, and then with the Muslim conquest of Spain, back to Europe, just as the printing press was invented.
- Euclid was printed soon after the Bible, in Latin translated from Persian translated from Greek.

Renewed attention to Euclid followed

- People still considered Euclid to be about actual space
- Even though "no part" and "breadthless length" are clearly not part of experience, they were considered as an *abstraction of experience*. There was just *one* possible notion of point, and one possible notion of line, and one possible notion of plane.
- A statement like Euclid 5, therefore, had to be true or false, and it was the job of the geometer to establish its truth on a sound logical basis, preferably from a simpler, more obvious axiom.

Failed attempts to prove Euclid 5

People tried to prove Euclid 5 by *reductio*, without success: Failed attempts were published by

- Simplicius (Byzantine, sixth century)
- al-Jawhari (Persian, ninth century)
- Nasir Eddin al Tusi
- Legendre (1752-1833), who continued to try to prove Euclid 5 until the year of his death, when he published a collection of his failed attempts, one of which repeated the erroneous assumption made by the above three.
- Lambert (whose wrong proof was only published posthumously)
- A Ph. D. thesis in 1763 found flaws in 28 different alleged proofs of Euclid 5.

An attempt to derive Euclid 5 from something simpler

Clairaut (1743) derived Euclid 5 from “rectangles exist”, which has the advantage over Euclid 5 that one (thinks one) can see a rectangle “before the eyes”, unlike the distant possible intersection point of the lines in Euclid 5.

Non-Euclidean geometry

The method of *reductio ad absurdum*, or for short just *reductio*, refers to proof by contradiction. Girolamo Sachheri tried to prove Euclid 5 this way.

- He made long deductions and was in some sense the creator of non-Euclidean geometry.
- But he did not, apparently, understand what he had done, i.e., he continued to believe that more of these deductions would eventually lead to a contradiction.
- He published his work, saying that it "vindicated Euclid" because the theorem that the angle sum of a triangle could be less than two right angles is "repugnant to the nature of the straight line."
- He died a month after publication.
- For the history of logic, the important point is that Sachheri, and everyone else, still believed geometry was about the one true universe, the space we live in, and logical reasoning just a tool for uncovering truths about that space.

Bolyai

Janos Bolyai also tried to prove Euclid 5 by reductio. His father, also a mathematician, tried to warn him:

> *You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone ....*
> *I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time.*

But Bolyai the younger was not deterred, and was perhaps the first to perceive the consistency of the negation of Euclid 5:

> *Out of nothing I have created a strange new universe. All that I have sent you previously is like a house of cards in comparison with a tower.*

We are skipping over the stories of Bolyai, Gauss, and Lobachevsky here to keep on track with the history of logic.

Non-Euclidean Geometry

The modern point of view is that both Euclid 5 and its negation are consistent with Euclid's first four postulates. This can be shown by exhibiting a model that satisfies Euclid 1-4 but not Euclid 5. The modern point of view is that both Euclid 5 and its negation are consistent with Euclid's first four postulates.

The picture shows the Poincaré model, in which lines are circular arcs meeting the unit circle at right angles (including diameters of the unit circle), and distance is defined by a certain formula so that the boundary is infinitely far from any interior point.
Credit for the first construction of such a model goes to Beltrami (1868). He was the first to really know that Bolyai's "strange new universe" really exists.

Henri Poincaré (1854–1912)

Geometry was the mother of logic

- Beltrami's result was the first “unprovability theorem” in history.
- It was, in its time, as controversial as Gödel's theorem in the 1930s.
- In the 1870s, certain Italian and German geometers began to write down proofs with a logical precision hitherto not achieved. (Pasch, Pieri, Veronese for example).
- Guiseppe Peano, whose axioms for number theory are famous, and will be a central tool in this course, invented the logical notation that is used today, I believe strongly influenced by his colleagues who worked out the details of geometry very carefully.

Cantor and the diagonal method

- Not every strand in the history of logic came from geometry. Georg Cantor is of interest to us because of his **diagonal method**, invented in about 1880. Probably you are familiar with his proof that the real numbers in $[0, 1]$ form an uncountable set, but on the next slide we will review it.
- The diagonal method lies at the heart of the uncomputability and unprovability results of Turing and Gödel that are the main content of this course.

Cantor's proof

Suppose we could list the numbers in $[0,1]$ as $s_1, s_2, \ldots$. Expand each one as an infinite repeating binary expansion (possibly ending with all 0s; but we do not allow expansions ending in all 1s:

$$
\begin{array}{l}
s_1 = 00000000000\ldots \\
s_2 = 11111111111\ldots \\
s_3 = 01010101010\ldots \\
s_4 = 10101010101\ldots \\
s_5 = 11010110101\ldots \\
s_6 = 00110110110\ldots \\
s_7 = 10001000100\ldots \\
s_8 = 00110011001\ldots \\
s_9 = 11001100110\ldots \\
s_{10} = 11011100101\ldots \\
s_{11} = 11010100100\ldots \\
\vdots \quad \vdots\vdots\vdots\vdots\vdots\vdots\vdots\vdots\vdots\vdots\vdots \ddots
\end{array}
$$

$$
\boxed{s = 10111010011\ldots}
$$

Then define the "diagonal number" s so that it differs from s_i in the j-th decimal place. Then s does not occur in the list $s_1, s_2, \ldots$, because it differs from s_j in the j-th decimal place.

Frege and Quantifiers

As you learned in your first logic course, modern logicians freely use the "quantifiers" $\forall$ and $\exists$.

- It may surprise you to learn that these concepts were not used until the very end of the nineteenth century.
- There is nothing like a quantifier in Euclid or Aristotle.
- They were invented by Frege, but he had an awkward notation that nobody else used.
- Others used $(x)A(x)$ for $\forall x A(x)$, until Kleene's book that is the textbook for this course!
- Mathematicians *still* do not make much use of quantifiers and do not follow the syntax that logicians use.

Skolem functions

- Skolem showed using "Skolem functions" that quantifiers can be eliminated if you are willing to introduce new function symbols. That technique was one key to Gödel's completeness theorem.
- Let's review that. Suppose you have an axiom about addition

$$\forall x \exists y \, (x + y = 0).$$

  Then you could introduce a symbol for the additive inverse, say $-x$, with the axiom

$$\forall x (x + (-x) = 0).$$

- This axiom could replace the one containing $\exists$, and exactly the same theorems in the language without the new symbol will be provable.
- We say that the new theory is "conservative over" or is a "conservative extension of" the old theory.

Skolem: Father of model theory

Skolem proved the L owenheim-Skolem theorem in 1920. (See the Wikipedia article for the discussion of whether Löwenheim proved it in 1915 or not.) Skolem looked at it model-theoretically; proof theory didn't exist. He proved that every model of a (countable) theory T in a first-order language has a countable submodel. Let's review the proof.

- First introduce Skolem functions. Then you have a theory with quantifier-free axioms (and a larger language), such that every model of the new theory is one of the old theory too, and vice-versa (using the axiom of choice).
- Now start with interpretations of the constants, and close up under your Skolem functions.
- Specifically, let B_0 contain the interpretations of the constants, and let B_{n+1} be B_n together with all values of Skolem functions on elements of B_n. That is countably many elements altogether; let the countable model B be the union.
- Since T is quantifier-free, B satisfies the axioms of T.

David Hilbert (1862–1943)

You'll see in the Wikipedia article the ubiquitous picture of Hilbert in his Panama hat, so I thought I would give you a less common picture here.

Hilbert and the Axiomatic Method

In 1899 Hilbert published a book *Grundlagen der Geometrie*, reporting on two decades of work. Hilbert tried to repair the defects that had been discovered in Euclid's reasoning.

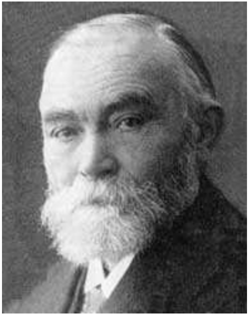
- Like Euclid, he used much work done by others.
- In particular Pasch was the first to introduce (1872) an important axiom that Euclid forgot.
- Hilbert's important contribution was the use of the axiomatic method. He said all the reasoning had to be correct, if throughout the work, you replace "point", "line", and "plane" by "table", "chair", and "beer mug."
- In other words, the reasoning must not refer to the meaning of the terms, but only to the assumptions and previous deductions and definitions.
- In today's terminology: the syntax must not depend on the semantics.

Importance of the axiomatic method

For mathematics this has great importance, as once a theory is developed this way, *it may have many models*.

- In these last decades of the nineteenth century, "abstract algebra" also developed, using the axiomatic method.
- Hilbert was one of the major developers of that field too.
- An understanding of the axiomatic method is fundamental in modern logic.
- Even physics works this way, since at least 1915. There are many "models" of Einstein's theory of general relativity, i.e., different possible solutions of his equations.
- Gödel even invented such a model in which the universe rotates and time travel is possible.

Friedrich Ludwig Gottlob Frege (1848-1925)



Frege and the comprehension axiom

Frege finished his masterpiece (whose name you will find as part of your homework), and sent a copy to Bertrand Russell in 1903.
Russell found a contradiction in Frege's axioms!
Today that contradiction is known as Russell's paradox, and is usually thought of as a contradiction in set theory, although Frege spoke not about sets but about "concepts".

- Intuitively a concept, or predicate, is something that is true or false of any object.
- In set-theoretical notation, $b \in X$ means b falls under the concept X, or belongs to the set X.
- This idea justifies the **comprehension axiom**:

$$\exists X \forall z \, (z \in X \leftrightarrow \phi(z))$$

  for each formula ϕ not containing X.
- It's customary to use a Skolem function for X, written

$$X = \{z : \ \phi(z)\}.$$

Russell's Paradox

Russell pointed out (by return mail) that

$$R = \{x : \; x \notin x\}$$

has, in contemporary terminology, serious issues.

- Namely, it leads a contradiction as soon as we ask whether $R \in R$ or not.

Russell had Volume I of Frege's work. His letter reached Frege as Volume II was about to be printed. Frege added an appendix saying this:

> *Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion.*

Russell and Whitehead

The situation appeared urgent to both Frege and Russell. Frege had no idea what to do about it. Russell consulted with his friend Alfred North Whitehead. They spent a dozen or so years developing axioms which they hoped would achieve these aims:

- The axioms should be strong enough to develop all of mathematics
- They should be free of contradictions
- Mathematics (including the numbers) would be defined in terms of logic, rather than the other way around. This philosophy is called "logicism."

They eventually published three volumes of dense, highly symbolic material, called *Principia Mathematica*, that are rarely read these days. It develops a "theory of types", in which one imagines sets of objects, sets of sets of objects, sets of those sets, and so on, in many "levels"–transfinitely many levels, in fact.

The trouble with *Principia*

- Each set has a certain level, and you use different variables for each level.
- So $\{x : \phi(x)\}$ has one level higher than the x in $\phi(x)$.
- Thus you can never form the paradoxical set R.
- There were, however, difficulties. For example, with this schema there will be real numbers of arbitrarily high level. So how can we ever form the set of all real numbers?
- We need to assume that all the real numbers will come in by some level. That was a consequence of the "axiom of reducibility" that Russell and Whitehead introduced.
- But, like Euclid 5, it was not considered evident enough to be an axiom!

The importance of *Principia*

Principia Mathematica was enormously influential, because it set the gold standard for formal logic.

- It showed by example what a formal system is, and what a formal proof is.
- Until then, these concepts had not been understood.
- Even by Hilbert, whose 1899 book on the Foundations of Geometry we have already mentioned.
- This example helped lead logicians to formulate the general concepts of "proof" and "theory" and "first-order language", even though *Principia* was not itself first-order.
- For example, when Gödel published the incompleteness theorem in 1931, it referred to unprovability in the system of PM, as the general concept of "theory" was not yet available.

Brouwer: Father of intuitionism

Another response to Russell's paradox came from L. E. J. Brouwer, a Dutch mathematician who had made a name for himself by proving several important theorems in topology. After doing so, he returned to the philosophical interests of his thesis, and founded the philosophy of "intuitionism", according to which

- Mathematics is based on mental constructions
- Symbols are used only as a communication tool, to enable you to make the mental constructions that I instruct you to make, and vice-versa.
- The rules of logic only reflect regularities in these constructions. They are more like observations than natural laws.
- Thus the formulas of logic or mathematics only summarize or describe constructions.

Intuitionistic logic

- In particular, if we prove $\exists x\, \phi(x)$, then we should provide a way to construct such an x. It isn't legitimate, in general, just to derive a contradiction from assuming no such x exists. Such a construction is of course some construction, but it's summarized by saying

$$\neg\neg\exists x\, \phi(x)$$

  which is weaker than $\exists x\, \phi(x)$.
- Brouwer himself had a low opinion of formal logic, but his student Heyting wrote down laws for intuitionistic logic.
- Brouwer strongly criticized the unrestricted use of the law of the excluded middle, which he claimed was unjustified. The reason is that $A \vee B$ is equivalent to asserting that there is an integer x such that if $x = 0$ then A, and if $x \neq 0$ then B. But if A is some unsolved problem and B is $\neg A$ then we have no idea how to construct x, so we are not entitled to assert $A \vee \neg A$.

Hilbert: Father of proof theory

Hilbert's reaction to Russell's paradox and the criticisms of Brouwer was to formulate a plan, known as "Hilbert's program", to settle these difficulties once and for all. The plan was this:

- Exhibit a formal theory in which mathematics can be formalized, the way mathematicians would like to do it.
- That theory might have many objectionable features (strong axioms, law of the excluded middle, etc.)
- But: it would be proved consistent! It would be rigorously demonstrated that no contradiction could be derived in that theory!
- Moreover, this consistency proof should be carried out by "finitistic means", so that *nobody* could object to the methods employed in the (small, safe) theory used for the consistency proof.

Hilbert and his assistants Bernays and Ackermann set out to do this. Of course, they encountered some difficulties, but in the process they wrote the first modern textbook on logic.

The *Entscheidungsproblem*

Hilbert and his co-authors formulated the first-order predicate calculus FOL, and the notion of a model of a theory, and asked the fundamental questions about these notions:

- is FOL complete? That is, does every consistent theory have a model?
- is FOL decidable? That is, is there an algorithm for deciding if a given formula is provable in FOL?

The second question was especially difficult, as the notion of "algorithm" had not yet been precisely defined, and the diagonal method appeared to be a serious obstacle to defining it.

- Gödel solved the first one in 1931 with his completeness theorem.
- Turing solve the second one in 1936, with his Turing machines.

The 1930s were the golden years of logic

- Turing invented Turing machines, solved the Entscheidungsproblem and proved that the halting problem is recursively unsolvable.
- Gödel proved the first incompleteness theorem: the system of *Principia*, if not contradictory, leaves some true theorems unprovable.
- Gödel proved the second incompleteness theorem: No sufficiently strong consistent formal theory can prove its own consistency. This was the death knell for Hilbert's program.
- Gödel developed the theory of general recursive functions
- Kleene developed the theory of partial recursive functions
- Church developed the λ-calculus
- These different notions of computability were shown to be equivalent.
- These results will be the (main) subject matter of this course

Kurt Gödel, age 20