Lecture 3: Peano Arithmetic, non-standard models, and Skolem's paradox

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Theories with foundational intent

Last time we discussed theories intended to have many models. There is another kind of logical theory, completely different in purpose. Namely:
Theories meant to **characterize** the fundamental structures of mathematics.
Here we list a few of these theories:

- Peano Arithmetic **PA**
- Zermelo Set Theory **Z**
- Zermelo-Frankel Set Theory **ZF**
- Russell and Whitehead's theory **PM** from *Principia Mathematica*

We start with Peano arithmetic **PA**.

Peano's axioms according to mathematicians

The point of Peano's axioms is to provide a definition, or perhaps you prefer the word "characterization", of the set of natural numbers $\mathbb{N}$ and the fundamental operations of addition and multiplication. Peano recognized that the most elementary operation on the natural numbers is "adding one", and he introduced the "successor" operation that takes x to the next natural number after x, which is denoted by $s(x)$ or by x'. Peano's axioms are:

- 0 is a natural number, and if x is a natural number so is x'.
- The natural numbers are closed under successor
- Successor is one-to-one, and 0 is not the successor of any natural number
- $\mathbb{N}$ is contained in every set that contains 0 and is closed under successor.

Addition and multiplication are defined recursively by
$a + 0 = a$ and $a + b' = (a + b)'$
$a \cdot 0 = 0$ and $a \cdot b' = a \cdot b + a$

Induction

The "induction axiom" is this one:
$\mathbb{N}$ is contained in every set that contains 0 and is closed under successor.
This axiom can be described as "$\mathbb{N}$ is the least set containing 0 and closed under successor". It can also be expressed as "$\mathbb{N}$ is the intersection of all sets containing 0 and closed under successor".
In class I will clarify its relation to the principle of mathematical induction that you learned in your mathematics classes.

Peano's success

Theorem

Up to isomorphism, there is exactly one model of Peano's axioms

Proof Sketch. Given a model M of Peano's axioms, an "initial segment up to n" is a subset Y of M containing 0, and containing n, and containing the successor of every element of Y but n. We consider functions mapping an initial segment of one model M onto initial segments of a second model N, and preserving successor. For each n in M, there is exactly one such function, as we see by considering the set of N for which this is true. That set contains 0 and is closed under successor. Then the union of such functions is a function φ mapping M into N, and closed under successor. The range of φ is a set containing the zero of N and closed under the successor function of N, so by the induction axiom, N is contained in the range of φ, making φ an isomorphism between M and N.

Thus Peano succeeded in his effort to characterize the natural numbers uniquely by simple axioms.

Induction and sets and FOL

As stated the induction axiom refers to *arbitrary sets*, since it says $\mathbb{N}$ is a subset of *ANY* set containing 0 and closed under successor.

- *Then it isn't a first-order statement*, unless the language has variables for sets and a predicate for membership.
- Does it make sense to define numbers in terms of sets? Numbers are more fundamental than sets, aren't they? We will leave this question aside and stick to questions with more precise answers.

Language of Peano Arithmetic

The theory called **PA** is a first-order theory, a kind of "approximation" to Peano's axioms.

The intended model of **PA** is the natural numbers $\mathbb{N}$ (which are $0, 1, 2, \ldots$), together with the usual operations of addition and multiplication, and the operation of successor. The successor of x is the next integer after x, written x'.
Correspondingly, the language has

- one sort of variables ("for the numbers"),
- two binary function symbols $+$ and $\cdot$,
- and one unary function symbol for successor.
- There is just one constant symbol, 0
- The language has a binary predicate $x = y$

Conventions about the language of **PA**

- Postfix notation x' is, in most books, not the official notation, which is something like $s(x)$ or $succ(x)$. When you see x', that is usually an "abbreviation" for the "official" formula.
- Of course it is a technicality: one can write the rules of FOL to permit postfix notation if desired.
- This is similar to the convention that we omit parentheses that seem superfluous or confusing to the human eye, even if they are officially required.
- Similarly, infix notation for $+$ and $\cdot$ is customary, as is the complete omission of the symbol for multiplication.
- $a \neq b$ abbreviates $\neg(a = b)$.

Some Axioms of **PA**

- $a' = b' \supset a = b$ (Successor is one-to-one)
- $a' \neq 0$ (Successor never takes the value 0)
- $a + 0 = a$ and $a + b' = (a + b)'$
- $a \cdot 0 = 0$ and $a \cdot b' = a \cdot b + a$

Kleene (page 82) includes some other axioms that would be automatically part of a theory in "FOL with equality". In class I will review the difference between FOL with equality and without. Kleene's official version of **PA** has the symbol for $=$ but not all of the equality axioms, because the rest can be proved.

The induction schema

A “schema” is an infinite family of formulas matching a particular form. The “induction schema” is

$$(A(0) \wedge \forall x\,(A(x) \supset A(x')) \supset \forall z\, A(z)$$

Here A can be any formula in the language of **PA**.
The first-order theory **PA** has for its axioms, those on the previous slide, plus the induction schema.

PA is a first-order theory

Its axioms are all (informally) consequences of Peano's (set-theoretical) axioms, if we believe the "arithmetic comprehension axiom", according to which, $\{x : \phi(x)\}$ exists for every formula ϕ in the language of **PA**. The phrase "arithmetical formula" is used as a synonym for "formula of **PA**."

Peano's characterization theorem fails for **PA**

By $\bar{n}$, we mean the term 0 with n successor symbols, so for example $\bar{2}$ is $0''$ and $\bar{4}$ is $0''''$.

Let us add a new constant to the language of **PA**, say c.

Let us also add the axioms $c \neq 0$ and $c \neq \bar{n}$ for $n = 1, 2, \ldots$.

Call this theory T.

- Any finite subset of T is consistent, by interpreting c as a large-enough integer, larger than any of the n occurring in axioms $c \neq \bar{n}$ in the finite subset of T.
- Therefore by the compactness theorem, T is consistent.
- Therefore by the completeness theorem, T has a model M.
- That model is not isomorphic to $\mathbb{N}$.

To prove this last claim, suppose that φ is an isomorphism from M to $\mathbb{N}$. Then φ takes the numerals of M (the elements interpreting the terms $\bar{n}$) onto the corresponding numerals of M. Let C be the member of M interpreting c. then $\varphi(C)$ cannot be equal to a numeral of $\mathbb{N}$. But *every* element of $\mathbb{N}$ is a numeral of $\mathbb{N}$, contradiction.

Two kinds of theories? No!

We said there are two kinds of theories:

- Those meant to have many models, such as group theory.
- Those meant to characterize a fundamental structure, such as $\mathbb{N}$ or the universe of sets.

But as we have seen, the latter goal cannot really be accomplished. A first-order theory *ALWAYS* has many models (if it has any infinite models).

- This is a consequence of the completeness theorem and the fact that proofs are finite (have only a finite number of symbols).
- There is no getting around it.
- There is only ONE kind of theory after all: the kind with many different models.
- This is the ultimate source of the difficulties further elaborated in the incompleteness theorems we will study in this course.

Zermelo Set Theory

Mathematicians are taught that all mathematics, or at least all known mathematics, can be derived from the basic axioms of set theory. Let's take a look at these axioms.

- The language is simple: just one binary relation $x \in y$ for set membership.
- Only one sort of objects: everything is a set.
- Two sets are equal if they have the same elements (axiom of extensionality):

$$(\forall x(x \in a \equiv x \in b)) \supset a = b.$$

- For simplicity, introduce a constant ϕ for the empty set.
- a binary function $\{x, y\}$ for the set whose members are x and y.
- $\{x\}$ abbreviates $\{x, x\}$.

Separation

- The **separation axiom schema** says, for each formula $A(x)$ not containing a,

$$\exists z \forall x \, (x \in z \equiv x \in a \wedge A(x)).$$

  That set z is abbreviated

$$\{x \in a : A(x)\}.$$

- Because of Russell's paradox we can't omit the part about $x \in a$.

Set Existence axioms

- the Axiom of the Union says that for every set x, there is a set containing all the elements of the elements of x. Thus if $x = \{A, B\}$ then the set of elements of elements of x is $A \cup B$, but more generally, x could be an infinite set.
- the Axiom of the Power Set says that for every set x, the set of all subsets of x exists.
- A copy of the natural numbers can be constructed as follows:

$$0 = \phi$$

$$n' = n \cup \{n\}$$

- The Axiom of Infinity says there exists a set containing ϕ and closed under that successor operation. (Zermelo's original axioms used $\{a\}$ for the successor of a but nowadays everyone uses the definition given here.)

Cantor's theorem in Zermelo

- Ordered pairs can be defined, for example

$$\langle a, b\rangle = \{\{a\}, \{a, b\}\}$$

- ω is the set of set-theoretic "integers".
- A *sequence* f is a set of ordered pairs of the form $\langle n, x\rangle$ such that

$$\langle n, x\rangle \in f \wedge \langle n, y\rangle \in f \supset x = y$$

$$\forall n \in \omega \exists y \left(\langle x, y \in f\right)$$

- A set is **countable** if it is the range of a sequence
- The power set of ω is not countable

Skolem's paradox

- Zermelo set theory can prove there is an uncountable set.
- But by the Löwenheim-Skolem theorem, Zermelo set theory has a countable model $\mathcal{M}$.
- Then the countable model $\mathcal{M}$ contains an uncountable set??

How can we resolve this "paradox"?

Resolution of Skolem's paradox

Let $\mathcal{M}$ be the countable model. Let Z be the element of $\mathcal{M}$ that plays the role of the power set of ω.

- Not every one of the subsets of ω is represented in the model, which contains only countably many subsets of ω.
- But also, the sequence that enumerates the elements of Z is not a member of $\mathcal{M}$.
- Although Z is (in fact) countable, it is not countable by means of any sequence represented in $\mathcal{M}$.
- Therefore, it satisfies the formula that says it is uncountable.

The contradiction is resolved by distinguishing between "countable" and "countable in $\mathcal{M}$", which means that $\mathcal{M}$ satisfies the formula expressing countability.

Nonstandard models of set theory

We can use the completeness theorem with Zermelo set theory just as well as we did with **PA**.

- In the context of set theory, $\bar{n}$ is a term denoting the integer n built up from the set-theoretic successor function.
- Add a new constant c and the axioms $c \in \omega$ and $c \neq \bar{n}$.
- Every finite subset of this theory is consistent.
- By the completeness theorem it has a model