Enneper's Surface

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1 Introduction

Enneper's surface was discovered in 1863 by Alfred Enneper, who was 33 at the time. This was seven years after his Ph. D. under the supervision of Dirichlet at Göttingen, where Enneper lived his entire life, from student to Professor Extraordinarius.

Enneper's surface is defined in the entire complex plane, so it is an example of a complete minimal surface (no boundary). However, we are interested in considering portions of it, defined in a disk of radius R. Then it is bounded by "Enneper's wire",

$$\Gamma_R(\theta) = \begin{bmatrix} R\cos\theta - \frac{1}{3}R^3\cos 3\theta \\ -R\sin\theta - \frac{1}{3}R^3\sin 3\theta \\ R^2\cos 2\theta \end{bmatrix}$$

The same formula, with r in place of R, defines Enneper's surface in polar coordinates.

2 Weierstrass representation

To show that Enneper's surface as defined above is indeed a minimal surface, we show that it arises from the Weierstrass representation if we take $f(z) = 1$ and $g(z) = z$. This gives us

$$X_z = \frac{1}{2}\begin{bmatrix} 1 - z^2 \\ i(1+z^2) \\ 2z \end{bmatrix} \tag{1}$$

Integrating, we have

$$\begin{aligned} X &= \mathrm{Re}\begin{bmatrix} z - \frac{1}{3}z^3 \\ i(z + \frac{1}{3}z^3) \\ z^2 \end{bmatrix} \\ &= \begin{bmatrix} r\cos\theta - \frac{1}{3}r^3\cos 3\theta \\ -r\sin\theta - \frac{1}{3}r^3\sin 3\theta \\ r^2\cos 2\theta \end{bmatrix} \end{aligned}$$

We now compute the unit normal in terms of the Weierstrass representation. We have (see [2] for details)

$$\begin{aligned} N &= \frac{X_x \times X_y}{|X_x \times X_y|} \\ &= \frac{1}{|g|^2+1}\begin{bmatrix} 2\,\mathrm{Re}\ g \\ 2\,\mathrm{Im}\ g \\ |g|^2 - 1 \end{bmatrix} \end{aligned}$$

For Enneper's surface we have $g(z) = z$, so

$$N = \frac{1}{r^2+1}\begin{bmatrix} 2x \\ 2y \\ r^2 - 1 \end{bmatrix}$$

$$N = \frac{1}{r^2+1}\begin{bmatrix} 2r\cos\theta \\ 2r\sin\theta \\ r^2-1 \end{bmatrix} \tag{2}$$

3 Non-parametric form

A minimal surface is said to be in non-parametric form if it is in the form $z = f(x, y)$. The following lemma gives a sufficient condition for a minimal surface to be expressible in this form.

Lemma 1 *Let X be a minimal surface bounded by a Jordan curve Γ. Suppose Γ projects one to one onto a curve γ in the xy plane. Suppose the unit normal N to X is never horizontal in the interior of its parameter domain. Then X is non-parametric over the interior of γ.*

Proof. The condition that N is nowhere horizontal implies, by the implicit function theorem, that the projection from X to the xy plane is locally invertible. Since the coordinate functions of a minimal surface are harmonic, that projection induces a locally conformal map from the parameter domain of X to the xy plane. The desired conclusion then follows from the following theorem.

Theorem 1 *Let f be a holomorphic map from the closed disk $\bar{B}$ onto $\bar{B}$, and suppose f is one to one on the boundary. Then f is a conformal map, i.e., there are no zeroes of the derivative f'.*

Remark. Radó [10] (p. 81) attributes this theorem to Darboux, with a reference to a German textbook on function theory that was unavailable in 1967 and is still unavailable in 2015. Darboux's theorem, as cited by Radó, only assumes continuity at the boundary rather than analyticity. In view of the apparent unavailability of Darboux's proof, we gave a proof of the analytic case in [4], which we do not repeat here.

Now we are ready to apply these results to Enneper's surface.

Lemma 2 *Let X be the portion of Enneper's surface defined in the disk of radius R. Then X is non-parametric if and only if $R \le 1$.*

Proof. Since the function g in the Weierstrass representation is the stereographic projection of the unit normal, and for Enneper's surface we have $g(z) = z$, the unit normal $N(z)$ to Enneper's surface is horizontal exactly when $|z| = 1$. Hence for $R \le 1$ the lemma above applies, so X is non-parametric if $R \le 1$. Conversely, if $R > 1$, one can show that there are vertical lines through Γ_R meeting it in two points. These equations are solved in detail in [4] (for another purpose). That completes the proof.

A famous sufficient condition for non-parametric form is given in the following theorem.

Theorem 2 (Radó) *Let the real analytic Jordan curve Γ have a monotonic strictly convex projection γ on the xy plane. Then any minimal surface bounded by Γ can be expressed in nonparametric form $z = f(x, y)$.*

Proof. Let X be a minimal surface bounded by Γ. By [7], X is real analytic up to the boundary. Suppose, for proof by contradiction, that X has a vertical tangent plane P at the point p. Then the intersection of P with (the range of) X consists of $2n$ real analytic arcs emanating from P at equal angles, for some $n \geq 2$. But since the projection of Γ is convex, P meets Γ in at most two points, contradiction. Hence X has no vertical tangent plane. Then by the previous theorem, X is expressible in non-parametric form over the interior of γ.

Remark. Radó's theorem is true without the assumption of real-analyticity, and with possibly non-strict convexity of the projection. Radó's own proof works without strict convexity, although he does not point that out. We do not give the proof under those weakened hypotheses.

3.1 Non-parametricity with respect to the xy plane

We investigate the projection of Enneper's wire on the xy plane. It turns out, the result is not as good as the one obtained in Lemma 2, in the sense that we only get $R \leq 1/\sqrt{3}$, but we get a stronger result, in that uniqueness follows.

Lemma 3 *For $R \leq \frac{1}{\sqrt{3}}$, the projection of Enneper's wire Γ_R on the xy plane is convex. Hence Γ_R bounds exactly one minimal surface.*

Proof. Let $\gamma = (X, Y)$ be the projection of Γ onto the xy plane. Then

$$\gamma(\theta) = \begin{bmatrix} R\cos\theta - \frac{1}{2}R^3\cos 3\theta \\ -R\sin\theta - \frac{1}{3}R^3\sin 3\theta \end{bmatrix}$$

The curvature of γ is given by

$$\frac{1}{(X_\theta^2 + Y_\theta^2)^{3/2}} \begin{bmatrix} X_\theta \\ Y_\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} X_{\theta\theta} \\ Y_{\theta\theta} \\ 0 \end{bmatrix}$$

which is zero exactly when

$$\begin{aligned} 0 &= \begin{bmatrix} X_\theta \\ Y_\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} X_{\theta\theta} \\ Y_{\theta\theta} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ X_\theta Y_{\theta\theta} - Y_\theta X_\theta \theta \end{bmatrix} \end{aligned}$$

Therefore we need to compute the least R such that for some θ we have $X_\theta Y_{\theta\theta} - Y_\theta X_{\theta\theta} = 0$.

Now we just compute.

$$\begin{aligned}
0 &= X_\theta Y_{\theta\theta} - Y_\theta X_{\theta\theta} \\
&= (-r\sin\theta + r^3 \sin 3\theta) Y_{\theta\theta} - (r\cos\theta + r^3\cos 3\theta) X_{\theta\theta} \\
&= (-R\sin\theta + R^3\sin 3\theta)(R\sin\theta + 3R^3 \sin 3\theta) - (-R\cos\theta - R^3\cos 3\theta)(-R\cos\theta + 3R^3\cos 3\theta)
\end{aligned}$$

Canceling R^2 from both sides we have

$$\begin{aligned}
0 &= (-\sin\theta + R^2\sin 3\theta)(\sin\theta + 3R^2\sin 3\theta) + (\cos\theta + R^2\cos 3\theta)(-\cos\theta + 3R^2\cos 3\theta) \\
&= 1 + 2R^2(-\sin 3\theta \sin\theta + \cos 3\theta\cos\theta) - 3R^4 \\
0 &= 1 + 2R^2 \cos 4\theta - 3R^4
\end{aligned}$$

For small R this expression is positive. It takes its minimum (for fixed R) when $\cos 4\theta = -1$. That minimum is $1 - 2R^2 - 3R^4$. This factors as $(1-3R^2)(1+R^2)$, so its only positive zero is $R = 1/\sqrt{3}$.

It follows that not only Enneper's surface, but *any* minimal surface bounded by Γ_R, is non-parametric, when $R \le 1/\sqrt{3}$. Since there is a maximum principle for the difference of two solutions of the non-parametric minimal surface equation, uniqueness follows. That completes the proof.

3.2 Non-parametricity relative to the yz plane

Let C be the projection of Enneper's wire on the yz plane, for $-\pi/2 < \theta < \pi/2$, plus the two horizontal line segments $y = \pm x$, $z = 0$ connecting Enneper's wire to the origin.

Lemma 4 *C bounds exactly one minimal surface, namely a portion of Enneper's surface, and it is non-parametric with respect to the yz plane.*

Proof. We use Radó's convex-projection theorem. However, in this case the projection contains a straight line, so it is not strictly convex. We consider the projection of C on the x_2x_3 plane; the projection of C is one to one, as required by the hypothesis of the convex-projection theorem, and we claim it is convex. The curved part is given by the equations

$$\gamma(\theta) = \begin{bmatrix} -R\sin\theta - \frac{R^3}{3}\sin 3\theta \\ R^2\cos 2\theta \end{bmatrix}$$

Here $-\pi/4 \le \theta \le \pi/4$. A plot of this curve for $R = 1.2$, and $-\pi/2 \le \theta < \pi/2$, is shown in Fig. 1. Note that the curve appears to be convex for θ in this range, not just in $[-\pi/4, \pi/4]$. The computation below confirms that result.

It suffices to show that the curvature vector of this curve always points inwards. Let $U(\theta)$ be the unit tangent to γ at $\gamma(\theta)$. Then $\kappa(\theta) = U_\theta$ and the inward normal N is obtained by rotating U by 90° counterclockwise. We must show that $\kappa \cdot N \ge 0$. This requires some computation, but it is considerably shorter than the proof that the total curvature of Enneper's wire is less than 6π. We do the first part of the computation in Sage, writing r for R and t for θ:

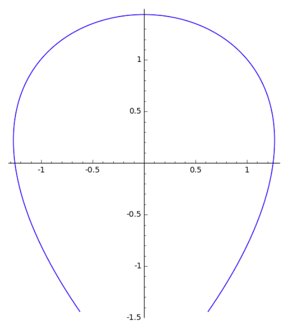

Figure 1: Part of the projection of Γ_R on the x_2x_3 plane, for $R = 1.2$. The curve is convex.

```
r,t = var('r,t')
assume(r > 0)
X = vector((r * cos(t) - (1/3)* r^3 * cos(3*t),
            - r * sin(t) - (1/3) *r^3 * sin(3*t),
             r^2 *cos(2*t)))
gamma = vector((X[1],X[2]))
U = gamma.diff(t)/abs(gamma.diff(t))
kap = U.diff(t)
norm = vector((-U[1],U[0]))
f = kap.dot_product(norm).trig_expand().trig_simplify()
```

The result of this (replacing t by θ)

$$f = \frac{3(4r^3 cos\theta \sin^2\theta + (r^3 + r)\cos\theta)}{16r^4\sin^4\theta + r^4 - 8(r^4 - r^2)\sin^2\theta + 2r^2 + 1} \tag{3}$$

Our aim is to show that $f \geq 0$. We begin by showing that the denominator is never zero. The denominator is a quadratic in $\sin^2\theta$. Writing it explicitly as such a quadratic with $s = \sin^2\theta$, it is

$$16r^4s^2 - 8(r^4 - r^2)s + (r^2 + 1)^2.$$

The discriminant of that quadratic is

$$\begin{aligned} D &= 64(r^4 - r^2)^2 - 64r^4(r^2 + 1)^2 \\ &= 64r^4((r^2 - 1)^2 - (r^2 + 1)^2) \\ &= 64r^4(-4r^2) \\ &= -256r^6 \\ &< 0 \end{aligned}$$

Since the discriminant is negative, the denominator of (3) is never zero, and hence always has one sign as a function of $\sin^2\theta$, for each r. When $\theta = 0$, the denominator is $(r^2+1)^2$, which is positive. Hence the denominator is always positive. Now consider the numerator, which is

$$4\cos\theta(4r^3\sin^2\theta + r^3 + r).$$

This is a positive factor times $\cos\theta$. But since $-\pi/4 \le \theta \le \pi/4$, we have $\cos\theta > 0$. In fact, as mentioned above, it suffices to assume $-\pi/2 < \theta < \pi/2$. Hence the numerator is positive. Since both the numerator and denominator of (3) are positive, f is positive. That completes the proof.

4 The second variation of D^2E

Consider the kernel equation for Enneper's surface in the disk of radius 1,

$$k_z X_z = 0$$

or in real form with $k = \psi X_\theta$,

$$X_\theta((\psi X_\theta)_r - (\psi X_r)_\theta) = 0.$$

We will show that $\psi = \sin(2\theta)$ solves this equation. On S^1 we have (even when $t \neq 0$)

$$\begin{aligned}
k &= \psi X_\theta \\
&= \sin 2\theta \,\mathrm{Re}\,(izX_z) \\
&= \frac{1}{2} i(z^2 - \bar{z}^2)\,\mathrm{Re}\begin{bmatrix} i(z - z^3) \\ -(z + z^3) \\ 2iz^2 \end{bmatrix} \\
&= \frac{1}{4} i(z^2 - \bar{z}^2)\begin{bmatrix} -iz^3 + iz - i\bar{z} + i\bar{z}^3 \\ -z^3 - z - \bar{z} - \bar{z}^3 \\ 2iz^2 - 2i\bar{z}^2 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} (z^2 - \bar{z}^2)(z^3 - z + \bar{z} - \bar{z}^3) \\ -i(z^2 - \bar{z}^2)(z^3 + z + \bar{z} + \bar{z}^3) \\ -(z^2 - \bar{z}^2)(2z^2 - 2\bar{z}^2) \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} z^5 - z^3 - \bar{z}^3 + \bar{z}^5 \\ -i(z^5 + z^3 - \bar{z}^3 - \bar{z}^5) \\ -2z^4 + 4 - 2\bar{z}^4 \end{bmatrix}
\end{aligned}$$

This expression for k is harmonic in the entire plane since evidently $\Delta k = k_{z\bar{z}} = 0$. Differentiating with respect to z we have (even when $t \neq 0$)

$$k_z = \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix} \tag{4}$$

Taking the dot product with X_z we have, when $t = 0$,

$$
\begin{aligned}
k_z X_z &= \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix} \cdot \begin{bmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ 5z^4 + 3z^2 \\ -8z^3 \end{bmatrix} \cdot \begin{bmatrix} 1 - z^2 \\ 1 + z^2 \\ 2z \end{bmatrix} \; = \; 0
\end{aligned}
$$

Similar calculations show that $\psi = a + b\cos\theta + c\sin\theta$ also yields a solution; this three-parameter family accounts for the conformal directions, and $\psi = 2\cos\theta$ represents a non-trivial kernel direction.

That k is the only kernel direction (orthogonal to the conformal directions) can be shown directly by writing a Fourier series for an unknown $\psi = \sum_{n=-\infty}^{\infty} a_n z^n$ on S^1 and showing ψ must have the form $a + b\cos\theta + c\sin\theta + d\sin 2\theta$, which is how we found $\psi = \sin 2\theta$ in the first place.

We also give a more informative proof that k is the only non-trivial kernel direction. The function $g(z)$ in the Weierstrass representation is the stereographic projection of the unit normal N, and for Enneper's surface $g(z) = z$. Hence, the Gaussian image of Enneper's surface in the unit disk is exactly the upper hemisphere. Hence the first eigenvalue of $D^2A(X)$ is 2, so the kernel of $D^2A(X)$ is one-dimensional, as the eigenspace of the least eigenvalue. But every member k of the kernel of $D^2E(x)$ gives rise to a member $\phi = k \cdot N$ of the kernel of $D^2A(x)$, and the map $k \mapsto \phi$ is one to one.

For $R < 1$, the Gaussian area of Enneper's surface over the disk of radius R is contained in a hemisphere, so the critical eigenvalue is more than 2 and the surface is a relative minimum of area. For $R > 1$, the Gaussian area contains a hemisphere, so the surface is not a relative minimum of area. Hence $R = 1$ is the only value for which the second variation has a kernel.

5 The third variation and fourth variations

5.1 The third variation

We now calculate the third variation of Enneper's surface (defined in the unit disk). We consider a variation $X(t)$ defined on S^1 by

$$X = X_0(e^{i(\theta + t\psi + O(t^2))})$$

where $\psi(\theta) = \sin 2\theta$ and the subscript in X_0 indicates $t = 0$. Differentiating with respect to t we have

$$X_t = (\psi + O(t))X_\theta.$$

Thus $k = X_t$ lies in the kernel of $D^2E(X_0)$ when $t = 0$.

Lemma 5 *The third variation of Enneper's surface is zero. Specifically, with the variation $X(t)$ given above, we have*

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 0.$$

Proof. We have (as shown in [3] following [12])

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 4\,\mathrm{Re} \int z k_z^2 \psi\, dz + 4\,\mathrm{Re} \int z(\psi k_\theta)_z X_z \psi\, dz \tag{5}$$

In the case of Enneper's surface, we have

$$\psi = \sin 2\theta = \frac{z^2 - z^{-2}}{2i}$$

where $z = e^{i\theta}$. By (4) we have $k_z = z^2 v$ for some holomorphic vector v. Hence

$$\psi k_z \qquad \text{is holomorphic in the unit disk.} \tag{6}$$

Therefore by Cauchy's theorem, the first integral in (5) is zero.

We now work on the second integral. We claim that also vanishes because the integrand is holomorphic in the disk. We have

$$k_\theta \quad = \quad 2\,\mathrm{Re}\,(izk_z)$$

Therefore

$$\begin{aligned} (\psi k_\theta)_z &= \psi 2\,\mathrm{Re}\,(izk_z) \\ &= 2\,\mathrm{Re}\,(iz\psi k_z) \end{aligned}$$

Since ψk_z is holomorphic, its complex derivative is half the complex derivative of its real part:

$$(\psi k_\theta)_z \quad = \quad (iz\psi k_z)_z$$

Putting that into the second term of (5) we have (since the first term is zero)

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} \quad = \quad 4\,\mathrm{Re} \int z(iz\psi k_z)_z X_z \psi\, dz$$

Integrating by parts we have

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} \quad = \quad -\,\mathrm{Re} \int (zX_z\psi)_z (iz\psi k_z)\, dz$$

But ψk_z is holomorphic, as shown in (6); and the z-derivative of any harmonic function is holomorphic, so $(zX_z\psi)_z$ is holomorphic. Then the entire integrand is holomorphic. Hence the integral is zero, by Cauchy's theorem. That completes the proof of the lemma.

5.2 The mixed third variations of Enneper's surface

We now consider the mixed third variation, where $k = \psi X_\theta$ is the kernel direction, and $h = \phi X_\theta$ is any tangent vector weakly orthogonal to the conformal directions.

Lemma 6 *Let X be the portion of Enneper's surface bounded by Enneper's wire with $R = 1$. Then the mixed third variations of X are zero. That is,*

$$D^2 E(X)[k, k, h] = 0$$

for any tangent vector h.

Proof. The proof divides into two cases: when h is a conformal direction, and when h is (weakly) orthogonal to all the conformal directions. Weakly orthogonal means that,

$$\langle\langle h, \ell \rangle\rangle := \int_0^{2\pi} h_r(e^{i\theta})\ell(e^{i\theta})\, d\theta = 0$$

for the three conformal directions ℓ. These two cases suffice to prove the lemma, since the third variation is a trilinear operator, so if k is the kernel direction, then any other tangent vector h can be written as $j + \ell$, where ℓ is a conformal direction, and h is weakly orthogonal to the conformal directions. Then we have

$$D^3 E(X)[k, k, h] = D^3 E(X)[k, k, j] + D^3 E(X)[k, k, \ell] = 0 + 0 = 0.$$

Since $E(\tilde{X})$ is constant along conformal orbits, we have $D^3 E(X)[k, k, \ell] = D^2 E(X)[k, k] = 0$, when $\ell = \phi X_\theta$ is a conformal direction, and $k = \psi X_\theta$ is the kernel direction, by considering the path

$$\tilde{(}X, t, s, \theta) = X(\theta + t\psi + s\phi).$$

Then

$$E(\tilde{X}) = D^2 E(X)[k, k] t^2$$

so taking the partial derivative with respect to s, we get 0; but this partial derivative is by definition the mixed variation $D^3 E(X)[k, k, \ell]$. That completes the proof in case h is a conformal direction.

Now we take up the second case. Let $h = \phi X_\theta$ be orthogonal to the conformal directions. Then we claim $\bar{z} h_z$ is holomorphic in the disk, i.e., h_z has no constant term, so h has no linear term. We prove this by calculating the weak inner product of h with the conformal directions. We have

$$\begin{aligned}
0 &= \langle\langle h, \ell \rangle\rangle \\
&= \int h_r \ell \, d\theta \\
&= \int 2\,\mathrm{Re}\,(z h_z) \ell \, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}\int_0^{2\pi} z h_z \ell \, d\theta \\
&= 2\,\mathrm{Re}\int_{S^1} z h_z \ell \, \frac{dz}{iz} \\
&= -2\,\mathrm{Re}\; i\int h_z \ell \, dz
\end{aligned}
$$

We note that X_z has a constant term (it suffices here to use the explicit formula (1), but that is consequence of the boundary being a Jordan curve). Then taking $\ell = 2\cos\theta X_\theta = 2\,\mathrm{Re}\,((z+\bar{z})zX_z)$, we find that

$$\mathrm{Re}\, i\int \bar{z}h_z \, dz = 0.$$

Taking $\ell = 2\sin\theta X_\theta$ we similarly find

$$\mathrm{Im}\, i\int \bar{z}h_z \, dz = 0.$$

Hence

$$\int \bar{z}h_z \, dz = 0$$

and by Cauchy's theorem, $\bar{z}h_z$ is holomorphic. Since $2\psi = -i(z^2 - \bar{z}^2)$, we have $2z\psi = -i(z^3 - \bar{z})$, and it follows that

$$z\psi h_z \qquad \text{is holomorphic} \tag{7}$$

Using (7), we can calculate the mixed third variation much as we did the one-directional third variation. Our starting point (see [3]) is the formula

$$D^3E(X)[k,k,h] \quad = \quad 4\,\mathrm{Re}\int z k_z h_z \psi \, dz + 4\,\mathrm{Re}\int z(\psi h_\theta)_z X_z \psi \, dz$$

For Enneper's surface, as shown above, ψk_z is holomorphic in the disk. Since the z-derivative of any harmonic function is holomorphic, the first term has a holomorphic integrand, so by Cauchy's theorem, it is zero. Hence

$$D^3E(X)[k,k,h] \quad = \quad 4\,\mathrm{Re}\int z(\psi h_\theta)_z X_z \psi \, dz$$

Expressing h_θ in complex form as $2\,\mathrm{Re}\,(izh_z)$ we have

$$D^3E(X)[k,k,h] \quad = \quad 4\,\mathrm{Re}\int (\,\mathrm{Re}\,(2z\psi h_z))_z (zX_z\psi) \, dz$$

Since $z\psi h_z$ is holomorphic, as we proved in (7), its complex derivative is twice the derivative of its real part, so we have

$$D^3E(X)[k,k,h] \quad = \quad 4\,\mathrm{Re}\int (z\psi h_z)_z (zX_z\psi) \, dz$$

Integrating by parts we have

$$D^3E(X)[k,k,h] \quad = \quad -4\,\mathrm{Re}\int (z\psi h_z)(zX_z\psi)_z\,dz$$

Since the z-derivative of any harmonic function is holomorphic, the second factor in the integrand is also holomorphic, so the whole integrand is holomorphic. Hence the value is 0, by Cauchy's theorem. That completes the proof.

Remark. It was necessary to integrate by parts because $z\psi X_z$ is not holomorphic; it contains a term in $\bar{z}$.

Remark. Ruchert [11] proved that Enneper's wire with $R \le 1$ bounds a unique minimal surface; hence Enneper's surface (defined in the disk of radius $R \le 1$) is a relative minimum of area (and hence also of E). It follows that all the mixed third variations $D^3E(X)[k,k,h]$ vanish, since otherwise we could construct a path $\tilde{X}$ along which the third derivative $E_{ttt}(\tilde{X})$ would be nonzero, and hence X would not be a relative minimum. Namely

$$\tilde{X}(t,\theta) = X(\theta + t\psi + t\phi)$$

where $k = \psi X_\theta$ is the kernel direction and $h = \phi X_\theta$ is any cokernel direction in which the mixed third variation is not zero. But as shown above, we can see by direct calculation that these mixed variations vanish; and indeed, we will also calculate the fourth variation and prove directly that Enneper's surface is a relative minimum when $R = 1$; and then we will use that fact to give a different proof of Ruchert's uniqueness theorem.

5.3 The fourth variation of Enneper's surface

We will compute the fourth variation of Enneper's surface along the path given by

$$X(t,\theta) = X_0(\theta + t\psi) \qquad \text{with } \psi = 2\sin 2\theta$$

We write $k = X_t = \psi X_\theta$. Since ψ does not depend on t, we have $k_t = \psi X_{\theta t} = \psi k_\theta$. The following formula for the fourth variation in a direction belonging to the kernel of the second variation is given in [3].

$$\begin{aligned} \left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} \quad &= \quad 8\,\mathrm{Re}\int zk_z k_{zt}\psi\,dz + 4\,\mathrm{Re}\int zk_{ttz}X_z\psi\,dz \\ &\quad + 12\,\mathrm{Re}\int zk_{zt}X_z\psi_t\,dz + 8\,\mathrm{Re}\int zk_z^2\psi_t\,dz \end{aligned}$$

Since we have assumed $\psi_t = 0$ the last two terms can be dropped:

$$\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} \quad = \quad 8\,\mathrm{Re}\int zk_z k_{zt}\psi\,dz + 4\,\mathrm{Re}\int zk_{ttz}X_z\psi\,dz$$

We have

$$\psi = \sin\theta = \frac{1}{2}(-iz^2 + iz^{-2}),$$

By (4), k_z is divisible by z^2. Hence $k_z\psi$ is holomorphic. Since the z-derivative of any harmonic function is holomorphic, and k_t is harmonic, so $k_{zt} = k_{tz}$ is holomorphic. Hence the first term also vanishes:

$$\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} = 4\,\mathrm{Re}\int z k_{ttz} X_z \psi\, dz$$

Recall from (4) that (even when $t \neq 0$)

$$k_z = \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix}$$

To use this equation when $t \neq 0$ we should put $z = e^{i(\theta + t\psi)}$, so we have

$$z_t = i\psi z$$

Differentiating k_z with respect to t we obtain

$$\begin{aligned}
k_{zt} &= \frac{1}{4}\frac{\partial}{\partial t}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix} \\
&= \frac{z_t}{4}\begin{bmatrix} 20z^3 - 6z \\ -20iz^3 - 6iz \\ -24z^2 \end{bmatrix} \\
&= \frac{i}{4}\psi z\begin{bmatrix} 20z^3 - 6z \\ -20iz^3 - 6iz \\ -24z^2 \end{bmatrix} \\
&= \frac{i}{4}dt\frac{z^2 - \bar{z}^2}{2i}z\begin{bmatrix} 20z^3 - 6z \\ -20iz^3 - 6iz \\ -24z^2 \end{bmatrix} \\
&= \frac{1}{8}\begin{bmatrix} 20z^6 - 6z^4 - 20z^2 + 6 \\ -20iz^6 - 6iz^4 + 20iz^2 + 6i \\ -24z^4 + 24 \end{bmatrix}
\end{aligned}$$

This came out holomorphic, as it had to, since it is also k_{tz} and k_t is harmonic. Now differentiate again with respect to t:

$$\begin{aligned}
k_{ztt} &= \frac{z_t}{8}\begin{bmatrix} 120z^5 - 24z^3 - 40z \\ -120iz^5 - 24iz^3 + 40iz \\ -96z^3 \end{bmatrix} \\
&= \frac{i\psi z}{8}\begin{bmatrix} 120z^5 - 24z^3 - 40z \\ -120iz^5 - 24iz^3 + 40iz \\ -96z^3 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{8}\frac{z^2-\bar{z}^2}{2i} z \begin{bmatrix} 120z^5 - 24z^3 - 40z \\ -120iz^5 - 24iz^3 + 40iz \\ -96z^3 \end{bmatrix} \\
&= \frac{1}{16} \begin{bmatrix} 120z^8 - 24z^6 - 160z^4 + 24z^2 + 40 \\ -120iz^8 - 24iz^6 - 80iz^4 + 24iz^2 - 40i \\ -96z^6 + 96z^2 \end{bmatrix}
\end{aligned}
$$

For Enneper's surface we have

$$
zX_z = \begin{bmatrix} z - z^3 \\ iz + iz^3 \\ 2z^2 \end{bmatrix}
$$

Taking the dot product with the previous equation, we have

$$
\begin{aligned}
zk_{ttz}X_z &= \frac{1}{16} \begin{bmatrix} 120z^8 - 24z^6 - 160z^4 + 24z^2 + 40 \\ -120iz^8 - 24iz^6 - 80iz^4 + 24iz^2 - 40i \\ -96z^6 + 96z^2 \end{bmatrix} \cdot \begin{bmatrix} z - z^3 \\ iz + iz^3 \\ 2z^2 \end{bmatrix} \\
&= 5z + O(z^2)
\end{aligned}
$$

Multiplying by ψ we have

$$
\begin{aligned}
zk_{ttz}X_z\psi &= (5z + O(z^2))\frac{z^2 - z^{-2}}{2i} \\
&= 5iz^{-1} + O(1)
\end{aligned}
$$

Integrating this around S^1, the $O(1)$ part is holomorphic, so it integrates to 0, and we have

$$
\begin{aligned}
\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} &= 4\,\mathrm{Re} \int zk_{ttz}X_z\psi\, dz \\
&= 4\,\mathrm{Re} \int \frac{5i}{z}\, dz \\
&= 4\,\mathrm{Re}\,\frac{5i}{2\pi i} \qquad \text{by Cauchy's residue theorem} \\
&= \frac{10}{\pi}
\end{aligned}
$$

We have proved

$$
\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} > 0 \tag{8}
$$

5.4 Relative minimum for $R = 1$

We need the following theorem, which is discussed in [3].

Theorem 3 *Let u be a minimal surface of disk type bounded by a Jordan curve Γ. Suppose that $D^2E(x)$ has a one-dimensional kernel (aside from the conformal directions) and that for some one-parameter family $X(t)$ of harmonic surfaces bounded by Γ, with $X(0) = u$ and $X_t(0) = k$ in the kernel of $D^2E(x)$, the third and fourth derivatives of $E(X(t))$ with respect to t are respectively zero and positive. Then u is a relative minimum of Dirichlet's energy.*

Corollary 1 *Enneper's surface for $R = 1$ is a relative minimum of area.*

Proof. Let u be Enneper's surface for $R = 1$, and let $\psi = \sin 2\theta$. Let $X(t, \theta) = X(\theta + t\psi)$. We have calculated the required second, third, and fourth derivatives of $E(X(t))$ in the previous sections, and they meet the hypotheses of the theorem. That completes the proof.

6 Curvature

6.1 Geodesic curvature of Enneper's wire

In this section we compute the geodesic curvature (with respect to Enneper's surface) of Enneper's wire Γ_R. We note that polar coordinates (r, θ) are not isothermal parameters for Enneper's surface; so we need $E = X_\theta^2$ and $G = X_r^2$ rather that just $W = \sqrt{EG}$. The element of geodesic curvature is given by a triple product:

$$\kappa_g ds \quad = \quad E^{-1}[X_\theta, X_{\theta\theta}, N]\, d\theta$$

where by definition $[a, b, c] = a \times b \cdot c$.

For Enneper's surface we have

$$X \quad = \quad \begin{bmatrix} r\cos\theta - \frac{1}{3}r^3\cos 3\theta \\ -r\sin\theta - \frac{1}{3}r^3\sin 3\theta \\ r^2\cos 2\theta \end{bmatrix}$$

Nitsche [9], p. 438, says "A straightforward computation yields the expressions pertaining to Enneper's surface," and gives the following results (with λ in place of θ):

$$\begin{aligned} E &= r^2(1+r^2)^2 \\ X \cdot N &= \frac{1}{3}r^2(3+r^2)(1+r^2)^{-1}\cos 2\lambda && (9) \\ E^{-1}[X_\theta, X_{\theta\theta}, N] &= (1+3r^2)(1+r^2)^{-1} && (10) \\ X \cdot X_r &= \frac{1}{3}r^2(1+r^2)(3+r^2) - \frac{2}{3}r^4\cos^2 2\lambda \end{aligned}$$

We used the mathematical software Sage to check these results. Here is the program we used:

```
r,t = var('r,t')
assume(r >= 0)
X = vector((r * cos(t) - (1/3)* r^3 * cos(3*t),
            - r * sin(t) - (1/3) *r^3 * sin(3*t),
             r^2 *cos(2*t)))
Xtheta = X.diff(t)
Xthetatheta = Xtheta.diff(t)
Xr = X.diff(r)
XrDotXtheta = Xr.dot_product(Xtheta).expand().trig_simplify()
assert(XrDotXtheta == 0)   ## If there were a typo this might fail
G = Xtheta.dot_product(Xtheta) .expand().trig_simplify().factor()
E = Xr.dot_product(Xr).expand().trig_simplify().factor()
Wsquared =  (E*G).expand().simplify().factor().simplify()
W = sqrt(Wsquared).canonicalize_radical().factor()
N =   Xr.cross_product(Xtheta)
for i in range(3):
N[i] = ((N[i].expand())/W).trig_simplify()
AbsN = abs(N).full_simplify()
assert(AbsN==1)
TripleProduct = Xtheta.cross_product(Xthetatheta).dot_product(N)
TripleProduct = TripleProduct.expand().trig_simplify().factor()
CurvatureElement = E^(-1)*TripleProduct
XdotXr = X.dot_product(Xr).expand().trig_simplify().factor()
XdotXr = XdotXr.reduce_trig().factor()
XdotN = X.dot_product(N).expand().trig_simplify().factor()
XdotN = XdotN.reduce_trig().factor()
print("%s %s" %("E = ",E))
print("%s %s" %("G = ",G))
print("%s %s" %("W = ",W))
print("%s %s" %("X . N  = ",XdotN))
print("%s %s" %("CurvatureElement = ",CurvatureElement))
print("%s %s" %("X . Xr  = ",XdotXr))
print("%s %s" %("G = ",G))
```

To run this program, cut and paste it into a file, and load (or attach) that file to a Sage terminal session. The output is

```
E =  (r^2 + 1)^2
W =  (r^2 + 1)^2*r
X . N  =  1/3*(r^2 + 3)*r^2*cos(2*t)/(r^2 + 1)
CurvatureElement =  (3*r^2 + 1)/(r^2 + 1)
X . Xr  =  1/3*(r^4 - r^2*cos(4*t) + 3*r^2 + 3)*r
```

The Sage program thus confirms Nitsche's calculations of E, the curvature element, and $X \cdot N$, but does not agree with his result for $X \cdot X_r$.[1] We do not need that result in this section anyway.

[1]The two results cannot be equal since for large r, Nitsche's result is asymptotic to r^6,

Now we have all the ingredients to calculate the geodesic curvature C_R of Enneper's wire Γ_R with respect to Enneper's surface:

$$
\begin{aligned}
C_R &= \int_0^{2\pi} \kappa_g \, ds \\
&= \int_0^{2\pi} E^{-1}[X_\theta, X_{\theta\theta}, N] \, d\theta \\
&= 2\pi \frac{3r^2+1}{r^2+1}
\end{aligned}
$$

as is confirmed by adding one line to the Sage program above:

```
GeodesicCurvature = integral(E^(-1)*TripleProduct,t,0,2*pi)
```

When $r = 1$ we have $C_1 = 4\pi$. The geodesic curvature is a monotonic function of r. To prove this, we show that its derivative is positive. Namely,

```
MonotonicTest = GeodesicCurvature.diff(r).full_simplify().factor()
```

produces

$$\frac{\partial}{\partial r} C_r = \frac{8\pi r}{(r^2+1)^2},$$

which is clearly positive.

It follows that for $R < 1$, Γ_R has geodesic curvature less than 4π, and hence, by the Gauss-Bonnet theorem and the theorem of Barbosa-do Carmo, the least eigenvalue of $D^2 A(X)$ (over the disk of radius R) is more than 2, so Enneper's surface is stable for $R < 1$.

6.2 Total curvature of Enneper's wire

Lemma 7 *The total curvature of Enneper's wire Γ_R is given by*

$$C_R = \frac{8R}{R^2+1} \frac{1}{u} \int_0^{\pi/2} \sqrt{1 - u^2 \sin^2 \theta} \, d\theta$$

where

Proof. The curvature of a curve Γ defined over the circle of radius R is given in terms of the unit tangent T_θ by

$$\kappa(\theta) = |T_\theta|$$

We have

$$T = \frac{\Gamma_\theta}{|\Gamma_\theta|}$$

while Sage's result is asymptotic to r^5. Since (for large r) we have $X = O(r^3)$, we have $X_r = O(r^2)$ and $X \cdot X_r = O(r^5)$, so Nitsche's result cannot be correct.

Finally, the total curvature is given by

$$C_R \quad = \quad \int_0^{2\pi} \kappa(\theta)\, d\theta$$

We compute κ using Sage as follows (the code repeats the definition of Enneper's wire, so it can be cut-and-pasted on a standalone basis):

```
r,t = var('r,t')
assume(r > 0)
X = vector((r * cos(t) - (1/3)* r^3 * cos(3*t),
            - r * sin(t) - (1/3) *r^3 * sin(3*t),
             r^2 *cos(2*t)))
Xtheta = X.diff(t)
AbsXtheta = abs(Xtheta)
T = (1/AbsXtheta) * Xtheta    ## unit tangent
kappa = abs(T.diff(t))
kappa = kappa.expand().trig_reduce().canonicalize_radical()
print(kappa)
```

(We recommend the reader who is not expert in sage to compare the results obtained with `trig_reduce`, `trig_simplify`, and `trig_expand` in the above script.) The output is

```
sqrt(9*r^4 + 2*r^2*cos(4*t) + 8*r^2 + 1)/(r^2 + 1)
```

or, properly typeset,

$$\kappa \quad = \quad \frac{1}{r^2+1}\sqrt{9r^4 + 2r^2\cos(4\theta) + 8r^2 + 1}$$

Write $\cos 4\theta$ as $1 - 2\sin^2 2\theta$:

$$\kappa \quad = \quad \frac{1}{r^2+1}\sqrt{9r^4 - 4r^2\sin^2 2\theta + 10r^2 + 1}$$

(I was not able to get Sage to take that step.) Put

$$u := 2r(1 + 10r^2 + 9r^4)^{-1/2}$$

Then

$$\kappa = \frac{2r}{r^2+1}\frac{1}{u}\sqrt{1 - u^2\sin^2\theta}$$

Integrating from 0 to 2π we obtain the desired total curvature. Since the integrand is a function of $\sin^2\theta$, we can change the interval of integration to $[0, 2\pi]$ if we multiply by 4. That completes the proof of the lemma.

The integral is $E(u)$, the complete elliptic integral of the second kind. Therefore, it can only be evaluated numerically. Adding the line

```
 Curvature = integral(kappa,t,0,2*pi)
```

to the Sage script above causes Maxima (the calculus engine of Sage) to crash, as of Sage version 6.5. So let us compute a table of its values. The following lines of Sage will do the job (tacked onto the above program for computing κ):

```
Curvature = lambda r : numerical_integral(kappa(r=r),0,2*pi)[0]
for i in range(1000):
R = 0.8 + 0.1*i
print("%lf %lf" % (R,Curvature(R)/pi))
```

The output is

```
0.800000 3.814841
0.900000 4.038869
1.000000 4.239358
1.100000 4.417379
1.200000 4.574703
1.300000 4.713388
1.400000 4.835538
1.500000 4.943161
1.600000 5.038096
1.700000 5.121989
2.000000 5.320974
12.000000 5.976998
22.000000 5.993126
32.000000 5.996748
42.000000 5.998111
52.000000 5.998768
62.000000 5.999133
72.000000 5.999357
82.000000 5.999504
92.000000 5.999606
```

Corollary 2 *The total curvature of Enneper's wire is a monotonically increasing function of R, whose limit at infinity is 6π.*

Proof. Since the integrand is bounded, we can take the limit of C_R as R goes to infinity straightforwardly; the integrand tends to 1 and the integral to $\pi/2$, while the part outside the integral tends to 12. Hence the limit is 6π as claimed. The monotonicity, however, is not so easy (and is not stated in [9]).

On the way to deriving Nitsche's formula for C_R given in the previous lemma, one finds this equation:

$$C_R = \int_0^{2\pi} \frac{\sqrt{9R^4 + 2R^2 \cos 4\theta + 8R^2 + 1}}{R^2 + 1}\, d\theta \tag{11}$$

This form seems slightly more convenient to work with. To prove the monotonicity, it suffices to show that the derivative of the integrand is positive. We have

$$
\begin{aligned}
&\frac{\partial}{\partial R}\left(\frac{\sqrt{9R^4+2R^2\cos 4\theta+8R^2+1}}{R^2+1}\right) \;=\\
&\frac{2\left(9R^3+R\cos 4\theta+4R\right)}{(R^2+1)\sqrt{9R^4+2\,R^2\cos 4\theta+8R^2+1}}-\frac{2R\sqrt{9R^4+2R^2\cos 4\theta+8R^2+1}}{(R^2+1)^2}
\end{aligned}
$$

and, as Nitsche would say, a direct computation shows that this is positive.

Here are a few lines of Sage code that produce (11).

```
r,t = var('r,t')
assume(r > 0)
X = vector((r * cos(t) - (1/3)* r^3 * cos(3*t),
            - r * sin(t) - (1/3) *r^3 * sin(3*t),
             r^2 *cos(2*t)))
Xtheta = X.diff(t)
AbsXtheta = abs(Xtheta)
T = (1/AbsXtheta) * Xtheta    ## unit tangent
kappa = abs(T.diff(t))
kappa = kappa.expand().trig_reduce().canonicalize_radical()
print(kappa)
```

6.3 The ellipsoid and the curvature vector

Lemma 8 *Enneper's wire lies on an ellipsoid E, and the curvature vector of Enneper's wire always points into E (and is never tangent to E).*

Proof. We write ${}^1\Gamma$, ${}^2\Gamma$, and ${}^3\Gamma$ for the components of $\Gamma = \Gamma_R$. To exhibit E, it suffices to find constants c and d such that

$$
{}^1\Gamma^2 + {}^2\Gamma^2 + c^2({}^3\Gamma)^2 = d^2.
$$

That is

$$
(R\cos\theta - \frac{1}{3}R^3\cos 3\theta)^2 + (-R\sin\theta - \frac{1}{3}R^3\sin 3\theta)^2 + c^2(R^2\cos 2\theta)^2 \quad = \quad d^2
$$

Simplifying, we have

$$
\begin{aligned}
R^2 + \frac{R^6}{9} + \frac{2R^4}{3}(\sin\theta\sin 3\theta - \cos\theta\cos 3\theta) + c^2R^4\cos^2 2\theta \quad &= \quad d^2\\
R^2 + \frac{R^6}{9} + \frac{2R^4}{3}\cos 2\theta + c^2R^4\cos^2 2\theta \quad &= \quad d^2
\end{aligned}
$$

This will work if $c^2 = 2/3$ and $d^2 = R^2 + R^6/9$. That completes the construction of E.

Next we compute the inward normal M to E at the point $\Gamma_R(\theta)$. That is just minus the gradient of the function on the left of the implicit equation of the ellipse, namely

$$-\nabla(x^2+y^2+\frac{2}{3}z^2) = \left[\begin{array}{c} -2x \\ -2y \\ -\frac{4}{3}z \end{array}\right].$$

We put in the components of Γ_R for x, y, and z, obtaining

$$M = \left[\begin{array}{l} -2R\cos\theta + \frac{2}{3}R^3\cos 3\theta \\ 2R\sin\theta + \frac{2}{3}R^3\sin 3\theta \\ -\frac{4}{3}R^2\cos 2\theta \end{array}\right].$$

Let κ be the curvature vector of Enneper's wire. We compute $\kappa \cdot M$ using the following Sage code (in which we write t for θ):

```
R,t = var('R,t')
X = vector((R * cos(t) - (1/3)* R^3 * cos(3*t),
           - R * sin(t) - (1/3) *R^3 * sin(3*t),
            R^2 *cos(2*t)))
Xtheta = X.diff(t)
T = 1/(abs(Xtheta)) * Xtheta   # unit tangent
kappavector = T.diff(t)
EllipsoidNormal = vector((-2*R*cos(t) + (2/3)*R^3*cos(3*t),
2*R*sin(t) + (2/3)*R^3 * sin(3*t),-(4/3)*R^2*cos(2*t)))
test = EllipsoidNormal.dot_product(kappavector)
test = test.trig_reduce().canonicalize_radical()
# that is much better than trig_simplify()!
print(test)
```

The result is

$$\kappa \cdot M \quad = \quad \left(\frac{2}{3}\right)\frac{3R^5 - 2R^3(3\cos 4\theta - 2) + 3R}{R^2+1}.$$

As a function of θ, this has its minimum when $\cos 4\theta = 1$. In that case the value of the numerator is $3R^5 - 2R^3 + 3R$. This is greater than $3R(R^4 - 2R^2 + 1) = 3R(R^2-1)^2 > 0$. It follows that $\kappa \cdot M$ is positive. That completes the proof of the lemma.

7 Enneper's wire and branch points

Theorem 4 (Rado, Nitsche, Meeks) *Enneper's wire Γ_R does not bound any branched minimal surface for $R \le \sqrt{3}$. More generally, Γ_R does not bound any minimal surface with self-intersections.*

Proof. By [6], Γ_R does not bound a minimal surface with a false branch point. For $R \leq 1$, Γ_R has a starlike projection. But that implies it cannot bound a minimal surface with a true branch point. See §91 and §384 of [9]. A minimal surface with a true branch point has lines of self-intersection emanating from the branch point [9], p. ?, so it suffices to show that Γ_R does not bound any minimal surface with self-intersections. That is true of any smooth Jordan curve lying on the boundary of a convex body, according to [8]. Enneper's wire lies on an ellipsoid, so it qualifies. That completes the proof.

8 Ruchert's uniqueness theorem

We follow [9], p. 437, in the computations of geodesic curvature and area, but we finish the proof in a different way. (In Nitsche's book, $\mathbf{X}$ is the unit normal, for which we use N.)

We start by considering an immersed minimal surface X bounded by a real-analytic Jordan curve Γ, and another immersed minimal surface Z bounded by the same curve. We suppose X and Z are defined on S^1 and we do not distinguish notational between these functions on S^1 and their harmonic extensions to the unit disk. Sometimes we write $X(\theta)$ instead of $X(e^{i\theta})$. Then for some real analytic periodic function λ with derivative $\lambda' > 0$, we have

$$Z(e^{i\theta}) = X(e^{i\lambda(\theta)}) = X(e^{i\lambda})$$

Then (following Nitsche in not writing the argument $e^{i\lambda}$ of X_θ) we have

$$\begin{aligned} Z_\theta &= \lambda' X_\theta \\ Z_{\theta\theta} &= (\lambda')^2 X_{\theta\theta} + \lambda'' X_\theta \end{aligned}$$

But Z_θ and X_θ are both tangent to Γ at the same point, so Z_r is perpendicular to X_θ. Therefore there are functions α and β such that

$$Z_r = \lambda'(\alpha X_r + \beta\sqrt{E}N)$$

where $E = |X_r|^2$ and $G = |X_\theta|^2$. (It is arbitrary which one of these we call E and which G; Nitsche is not explicit but it seems this is his choice.) Since $|Z_\theta|^2 = (\lambda')^2 G$, and $|Z_r|^2 = (1/r^2)|Z_\theta|^2 = (\lambda')^2 G/r^2$, but also

$$|Z_r|^2 = (\lambda')^2(\alpha^2 + \beta^2)E,$$

and $E = G/r^2$, we have $\alpha^2 + \beta^2 = 1$.

The unit normal $\tilde{N}$ to Z is given (on S^1) by

$$\begin{aligned} \tilde{N} &= \frac{Z_r \times Z_\theta}{|Z_r||Z_\theta|} \\ &= \frac{\lambda'(\alpha X_r + \beta\sqrt{E}N) \times \lambda' X_\theta}{(\lambda')^2 W(\alpha^2 + \beta^2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda'(\alpha X_r + \beta\sqrt{E}N)\times\lambda' X_\theta}{(\lambda')^2 W} \qquad \text{since } \alpha^2+\beta^2=1. \\
&= \frac{(\alpha X_r + \beta\sqrt{E}N)\times X_\theta}{W} \\
&= \alpha N - \frac{\beta X_r}{\sqrt{E}} \qquad \text{since } W^2 = EG
\end{aligned}
$$

Next we calculate the geodesic curvature $\tilde{\kappa}_g$ of Γ with respect to the surface Z. That is given (see [9] §378) by the triple product

$$
\tilde{\kappa}_g \tilde{E} d\tilde{s} \;=\; [Z_\theta, Z_{\theta\theta}, \tilde{N}]d\theta
$$

where $\tilde{E} = |Z_r|^2 = (\lambda')^2 E$. Then

$$
\begin{aligned}
\tilde{\kappa}_g(\lambda')^2 E\, d\tilde{s} &= [Z_\theta, Z_{\theta\theta}, \tilde{N}]d\theta \\
&= \left[\lambda' X_\theta, (\lambda')^2 X_{\theta\theta} + \lambda'' X_\theta, \alpha N - \frac{\beta X_r}{\sqrt{E}}\right] d\theta
\end{aligned}
$$

Since a triple product with two adjacent terms vanishes, the term with $\lambda'' X_\theta$ drops out, leaving

$$
\tilde{\kappa}_g(\lambda')^2 E\, d\tilde{s} \;=\; \left[\lambda' X_\theta, (\lambda')^2 X_{\theta\theta}, \alpha N - \frac{\beta X_r}{\sqrt{E}}\right] d\theta
$$

Dividing by $E(\lambda')^2$ we have

$$
\tilde{\kappa}_g\, d\tilde{s} \;=\; \alpha\lambda' E^{-1}[X_\theta, X_{\theta\theta}, N]\, d\theta - \frac{\beta}{E^{3/2}}\lambda'[X_\theta, X_{\theta\theta}, X_r]\, d\theta \tag{12}
$$

Splitting $X_{\theta\theta}$ into normal, radial, and tangential components, we have

$$
X_{\theta\theta} = (X_{\theta\theta}\cdot N)N + (X_{\theta\theta}\cdot X_\theta)X_\theta + (X_{\theta\theta}\cdot X_r)X_r.
$$

Putting that into the triple product, and remembering that a triple product with identical adjacent terms is zero, the terms in X_θ and X_r vanish, leaving

$$
\begin{aligned}
[X_\theta, X_{\theta\theta}, X_r] &= [X_\theta, N, X_r](X_{\theta\theta}\cdot N) \\
&= (X_\theta \times N \cdot X_r)(X_{\theta\theta}\cdot N)
\end{aligned}
$$

Now $X_\theta \times N$ is in the direction of X_r, with magnitude $\sqrt{G}$. So $X_\theta \times N \cdot X_r = \sqrt{EG} = \sqrt{W}$. Hence

$$
\begin{aligned}
[X_\theta, X_{\theta\theta}, X_r] &= \sqrt{W}(X_{\theta\theta}\cdot N) \\
&= \sqrt{W}(X_{\theta\theta}\cdot N) \qquad \text{since } X_r\cdot X_r = G \\
&= -\sqrt{W} X_\theta N_\theta \qquad \text{since } (X_\theta\cdot N)_\theta = 0
\end{aligned}
$$

Putting that result into (12) we have

$$
\begin{aligned}
\tilde{\kappa}_g\, d\tilde{s} &= \alpha\lambda' E^{-1}[X_\theta, X_{\theta\theta}, N]\, d\theta + \beta W^{1/2} E^{-3/2}\lambda' X_\theta N_\theta\, d\theta \\
&= \alpha\lambda' E^{-1}[X_\theta, X_{\theta\theta}, N]\, d\theta + \beta G^{1/2} E^{-1}\lambda' X_\theta N_\theta\, d\theta \qquad (13)
\end{aligned}
$$

So far in this section, we have not used the fact that X is Enneper's surface. It could be any minimal surface, so far. But now, we work out $X_\theta N_\theta$ for Enneper's surface.

$$
\begin{aligned}
X_\theta &= \begin{bmatrix} -r\sin\theta + r^3\sin 3\theta \\ -r\cos\theta - r^3\cos 3\theta \\ -2r^2\cos 2\theta \end{bmatrix} \\
N &= \frac{1}{r^2+1}\begin{bmatrix} 2r\cos\theta \\ 2r\sin\theta \\ r^2-1 \end{bmatrix} \qquad \text{by (2)} \\
N_\theta &= \frac{1}{r^2+1}\begin{bmatrix} -2r\sin\theta \\ 2r\cos\theta \\ 0 \end{bmatrix} \\
X_\theta N_\theta &= \frac{1}{r^2+1}\begin{bmatrix} -r\sin\theta + r^3\sin 3\theta \\ -r\cos\theta - r^3\cos 3\theta \\ -2r^2\cos 2\theta \end{bmatrix} \cdot \begin{bmatrix} -2r\sin\theta \\ 2r\cos\theta \\ 0 \end{bmatrix} \\
&= -2r^2\cos(2\theta)
\end{aligned}
$$

as can be confirmed by the following Sage code:

```
XthetaDotNtheta = Xtheta.dot_product(Ntheta).expand()
XthetaDotNtheta = XthetaDotNtheta.trig_simplify().factor().trig_reduce()
print("%s %s" %("Xtheta . Ntheta  = ",XthetaDotNtheta))
```

Putting that value of $X_\theta N_\theta$ into (13), with λ substituted for θ, we have

$$\tilde{\kappa}_g\, d\tilde{s} \;=\; \alpha\lambda' E^{-1}[X_\theta, X_{\theta\theta}, N]\, d\theta - \beta E^{-1}\sqrt{G}\lambda' 2r^2\cos 2\lambda\, d\theta$$

Putting in the value of $E^{-1}[X_\theta, X_{\theta\theta}, N]$ from (10), we have

$$\tilde{\kappa}_g\, d\tilde{s} \;=\; \alpha\lambda'\frac{1+3r^2}{1+r^2}\, d\theta - \beta E^{-1}\sqrt{G}\lambda' 2r^2\cos 2\lambda\, d\theta$$

Since $G = (r^2+1)^2$ and $E = r^2 G$, we have $E^{-1}\sqrt{G} = 1/(r^2(r^2+1))$. Hence

$$\tilde{\kappa}_g\, d\tilde{s} \;=\; \alpha\lambda'\frac{1+3r^2}{1+r^2}\, d\theta - \beta\lambda'\frac{2\cos 2\lambda}{r^2+1}\, d\theta \tag{14}$$

which, miraculously, is exactly the equation obtained by Nitsche in the middle of p. 438 (just after the incorrect equation mentioned above).

Now, Ruchert's main contribution is the following lemma:

Lemma 9 (Ruchert) *Let $\tilde{X}$ be any immersed minimal surface bounded by Enneper's wire Γ_R different from Enneper's surface X. Then either $\tilde{X}$ has smaller Dirichlet energy than X, or the geodesic curvature of Γ_R is smaller with respect to $\tilde{X}$ than with respect to X.*

Remarks. Since these are minimal surfaces, their areas are the same as their Dirichlet energies. The proof uses two facts about Enneper's surface: (i) $X \cdot X_r > 0$, and (ii) XN and $X_\theta N_\theta$ have opposite signs and the same θ-dependence.

Proof. Since we have used E for X_r^2, we follow Nitsche in temporarily using I for Dirichlet's energy. We have

$$\begin{aligned}
2I(X) &= \int_0^{2\pi} X_r(\lambda(\theta))X(\lambda(\theta))\,\lambda'\,d\theta \\
2I(\tilde{X}) &= \int_0^{2\pi} \tilde{X}_r\tilde{X}\,d\theta \\
&= \int_0^{2\pi} \lambda'[\alpha X_r + \beta E^{1/2}N]X(\lambda(\theta))\,d\theta
\end{aligned}$$

Subtracting, we have

$$\begin{aligned}
2I(X) - 2I(\tilde{X}) &= \int_0^{2\pi} [(1-\alpha)X_rX - \beta E^{1/2}XN]\lambda'\,d\theta \\
&= \int_0^{2\pi} (1-\alpha)X_rX\lambda'\,d\theta - \int_0^{2\pi} \beta E^{1/2}XN\lambda'\,d\theta
\end{aligned}$$

We have already computed E and XN at the end of §6.1. Using those results, we have

$$\begin{aligned}
E^{1/2}XN &= (r^2+1)r\big(\frac{1}{3}\big)\frac{r^2(r^2+3)}{r^2+1}\cos 2\lambda \\
&= \frac{1}{3}r^3(r^2+3)\cos 2\lambda
\end{aligned}$$

Hence

$$2I(X) - 2I(\tilde{X}) = \int_0^{2\pi} (1-\alpha)X_r(e^{i\lambda})X(e^{i\lambda})\lambda'\,d\theta - \frac{1}{3}r^3(r^2+3)\int_0^{2\pi} \beta\cos(2\lambda)\lambda'\,d\theta$$

Integrating (14) to find the geodesic curvature $\tilde{C}$ of $\tilde{X}$, we have

$$\tilde{C} = \frac{1+3r^2}{1+r^2}\int_0^{2\pi} \alpha\lambda'\,d\theta - \frac{2}{r^2+1}\int_0^{2\pi} \beta\cos(2\lambda)\lambda'\,d\theta$$

Now the sign of

$$J := \int_0^{2\pi} \beta\cos(2\lambda)\lambda'\,d\theta$$

is crucial. First suppose $J \ge 0$. Then we have

$$\tilde{C} \le \frac{1+3r^2}{1+r^2}\int_0^{2\pi} \alpha\lambda'\,d\theta$$

Since $\alpha \leq 1$ and λ is periodic and monotone, we have

$$\begin{aligned} \tilde{C} &\leq \frac{1+3r^2}{1+r^2}\int_0^{2\pi} \lambda'\, d\theta \\ &= \frac{1+3r^2}{1+r^2} 2\pi \\ &= C_r \qquad \text{the geodesic curvature with respect to Enneper's surface} \end{aligned}$$

Equality can hold only if α is identically zero, in which case $\tilde{X} = X$. That completes the proof in case $J \geq 0$. Now suppose $J < 0$. Then we have

$$I(X) - I(\tilde{X}) \quad > \quad \frac{1}{2}\int_0^{2\pi} (1-\alpha) X_r(e^{i\lambda}) X(e^{i\lambda})\lambda'\, d\theta$$

Since $\alpha^2 + \beta^2 = 1$, we have $\alpha^2 \leq 1$ and hence $1 + \alpha \geq 0$. Then (unless $\alpha = -1$)

$$1 - \alpha = \frac{1-\alpha^2}{1+\alpha} \geq 0.$$

Hence $\alpha \leq 1$. We will show by direct computation that $X_r \cdot X > 0$. Then the entire integrand is positive. We have (see the output of the Sage program in §6.1)

$$X_r \cdot X = \frac{1}{3} r(r^4 + r^2(3 - \cos 4\lambda) + 3)$$

Since $|\cos 4\lambda| \leq 1$ the right side is positive. That completes the proof of the lemma.

Theorem 5 (Ruchert) *Enneper's wire Γ_R bounds exactly one minimal surface (namely Enneper's surface) for $R \leq 1$.*

Proof. Let $R \leq 1$ and suppose, for proof by contradiction, that $\tilde{X}$ is a minimal surface bounded by Γ_R and different from Enneper's surface. By Theorem 4, $\tilde{X}$ is immersed. As calculated in §6.1, the geodesic curvature of Γ_R with respect to Enneper's surface is monontonically increasing with R and has the value 4π when $R = 1$. By Ruchert's lemma, $\tilde{X}$ either has smaller area than Enneper's surface, or Γ_R has geodesic curvature less than 4π.

We know that X is a relative minimum of area, as shown by calculating its fourth variation. Using this fact we can prove Ruchert's theorem using the mountain-pass theorem (see Chapter 6 of [5]), which is simpler than the proof in Nitsche's book.

Case 1: $\tilde{X}$ has smaller area than Enneper's surface X. Then, replacing $\tilde{X}$ by another minimal surface if necessary, we may assume $\tilde{X}$ is an absolute minimum of area, and different from X. since both X and $\tilde{X}$ are relative minima, we can apply the mountain-pass theorem to conclude that Γ_R also bounds an unstable minimal surface Z, with area greater than the areas of X or $\tilde{X}$. By Theorem 4, Z is immersed. Since the area of Z is greater than that of X, by Ruchert's lemma, Γ_R has geodesic curvature less than 4π. Hence, by the Gauss-Bonnet theorem, its Gaussian area is less than 2π. Then by the theorem of Barbosa-do Carmo, Z is stable, contradiction. That completes the proof in case 1.

Case 2: The geodesic curvature of Γ_R with respect to $\tilde{X}$ is less than 4π. Then, by Gauss-Bonnet and Barbosa-do Carmo, $\tilde{X}$ has positive second variation, and hence is a relative minimum. Then again using the mountain pass theorem, we find an unstable minimal surface Z bounded by Γ_R, and proceed as in Case 1. That completes the proof of Ruchert's theorem.

9 $D^2A(X)$ and $D^2E(X)$

9.1 Correspondence between the kernel functions

There is a connection between the kernels of the second variation of Dirichlet's energy and area. Namely, if k belongs to the kernel of $D^2E(X)$, then $\phi = k \cdot N$ belongs to the kernel of $D^2A(X)$, i.e., $\Delta\phi - 2KW\phi = 0$ in the parameter domain and $\phi = 0$ on the boundary.

Conversely, if ϕ belongs to the kernel of $D^2A(X)$, then there is a real-analytic, complex-valued function h defined in the disk, such that

$$k := \mathrm{Re}\,(hX_z) + \phi N$$

and k is in the kernel of $D^2E(X)$. Because k on the boundary is a scalar multiple of X_θ, we have $\mathrm{Re}\,(\bar{z}h) = 0$ on S^1. The equation determining h is

$$h_{\bar{z}} = \phi X_{\bar{z}\bar{z}} \frac{N}{W}.$$

Here $W = |X_z|^2$. These results were first proved in [1] and the proof is easily available on the Web in [2].

Note that polar coordinates are not isothermal; if we calculate $E = X_r^2$, $G = X_\theta^2$, and $W = \sqrt{EG}$ then this is *not* the W that we need to use in the equation for h, as the equation for h assumes X is given in harmonic isothermal parameters. Therefore if X is given in polar coordinates, then we need to use $E = X_r^2$ as W in the equation for h, as this is equal to $|X_z|^2$.

A purely real version of the result says that there are functions ψ and χ defined in the unit disk such that

$$k = \psi X_\theta + \chi X_r + \phi N.$$

Here

$$\psi = \mathrm{Im}\,(\bar{z}h)$$
$$\chi = \mathrm{Re}\,(\bar{z}h)$$

Note that ψ is defined in the whole unit disk, not just on S^1; since k is a tangent vector, we have $k = \psi X_\theta$ on S^1, so χ and ϕ are zero on S^1.

9.2 How this works out for Enneper's surface

As an exercise, we work out what h, ψ, χ, and ϕ are for Enneper's surface (defined in the unit disk). First, the eigenfunction ϕ is computed easily, since it is just ${}^3N$, the third component of the unit normal, as on the sphere $\Delta N = N$, with Δ the Laplace-Beltrami operator of the sphere. That gives us (referring to the formula for the normal in terms of the Weierstrass representation)

$$\phi = \frac{r^2-1}{r^2+1}.$$

For Enneper's surface we have $E = X_r^2 = (r^2+1)^2$. As explained above, we need to use E for W in the equation for h, as it is E that is equal to $|X_z|^2$. We have

$$N = \frac{1}{r^2+1}\begin{bmatrix} 2x \\ 2y \\ r^2-1 \end{bmatrix}$$

$$X_z = \frac{1}{2}\begin{bmatrix} 1-z^2 \\ i(1+z^2) \\ 2z \end{bmatrix}$$

Hence

$$X_{\bar z} = \frac{1}{2}\begin{bmatrix} 1-\bar z^2 \\ -i(1+\bar z^2) \\ 2\bar z \end{bmatrix}$$

Differentiating, we have

$$X_{\bar z\bar z} = \begin{bmatrix} -\bar z \\ -i\bar z \\ 2 \end{bmatrix}$$

Hence the equation for h is

$$h_{\bar z} = \phi X_{\bar z\bar z}\frac{N}{W} = \frac{r^2-1}{(r^2+1)^4}\begin{bmatrix} -\bar z \\ -i\bar z \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ r^2-1 \end{bmatrix}$$

That is

$$\begin{aligned} h_{\bar z} &= \frac{r^2-1}{(r^2+1)^4}(-2\bar z(x+iy) + 2(r^2+1)) \\ &= \frac{r^2-1}{(r^2+1)^4}(-2\bar z z + 2(r^2+1)) \\ &= \frac{r^2-1}{(r^2+1)^4}(-2r^2 + 2(r^2+1)) \\ &= \frac{r^2-1}{(r^2+1)^4} \end{aligned}$$

Now, to solve this equation for h, we first solve the equation

$$\Delta V = \frac{r^2 - 1}{(r^2 + 1)^4}. \tag{15}$$

and then set $h = V_z + A$, where A is a holomorphic function in the disk (which will be chosen to fix the boundary values). Then $h_{\bar{z}} = V_{z\bar{z}} = \Delta V$ and we have the desired equation. Then

$$\begin{aligned} \chi - i\psi &= \bar{z}h \\ &= \bar{z}(V_z + A) \\ &= \frac{\bar{z}^2}{r^2} z(V_z + A) \\ &= e^{2i\theta} z(V_z + A) \\ &= e^{2i\theta}(rV_r - iV_\theta + zA) \end{aligned}$$

Since ΔV depends only on r, we can assume $V_\theta = 0$ (as we will show below). Then

$$\begin{aligned} \chi - i\psi &= (\cos 2\theta - i \sin 2\theta) r V_r + (\cos\theta + i\sin\theta) A \\ \chi &= \cos 2\theta V_r + \cos\theta \operatorname{Re} A - \sin\theta \operatorname{Im} A \\ \psi &= \sin 2\theta V_r - \sin\theta \operatorname{Re} A - \cos\theta \operatorname{Im} A \end{aligned}$$

So we have to take $A = -cz$ where $c = V_r|_{r=1}$ to make χ be zero on S^1:

$$\begin{aligned} \chi &= \cos 2\theta (V_r - cr) \\ \psi &= \sin 2\theta (V_r + cr) \end{aligned}$$

It remains to find V by solving (15). Since the right side does not depend on θ, we look for a solution $V(r)$. Then $\Delta V = V_{rr} + (1/r)V_r$, so we have to solve the ordinary linear differential equation

$$V'' + \frac{1}{r}V' - \frac{r^2 - 1}{r(r^2 + 1)^4} = 0.$$

We want a solution with $V_r = 0$ when $r = 0$. The general solution of this equation can be found, at least in theory, using this piece of Sage code:

```
V = function('V',r)
de = diff(V,r,2) + (1/r)*diff(V,r) - (r^2-1)/((r^2+1)^4) == 0
v = desolve(de,V)
```

The result is (equivalent to)

$$V = K_2 - \frac{1}{12}\log r + K_1 + \frac{r^2 + 3}{24(r^2 + 1)^2} - \frac{1}{24}\log(r^2 + 1)$$

The constant K_2 must be $1/12$ to prevent a singularity at the origin. Hence

$$V = K_1 + \frac{r^2+3}{24(r^2+1)^2} - \frac{1}{24}\log(r^2+1)$$

Differentiating, we have

$$V_r = -\frac{1}{12}\frac{r^3}{(r^2+1)^2} - \frac{1}{6}\frac{r(r^2+3)}{(r^2+1)^3}$$

When $r = 1$ we find $c = V_r|_{r=1} = -5/48$. Hence

$$\begin{aligned}\chi &= \left(-\frac{1}{12}\frac{r^3}{(r^2+1)^2} - \frac{1}{6}\frac{r(r^2+3)}{(r^2+1)^3} + \frac{5r}{48}\right)\cos 2\theta \\ \psi &= \left(-\frac{1}{12}\frac{r^3}{(r^2+1)^2} - \frac{1}{6}\frac{r(r^2+3)}{(r^2+1)^3} - \frac{5r}{48}\right)\sin 2\theta\end{aligned}$$

Note that the constant K_1 does not affect χ and ψ. Although k, ψ, χ, and ϕ can be multiplied by the same arbitrary constant, we fixed that constant when we normalized ϕ when setting up the equation for h. That everything is in order is evidenced by the facts that on S^1, we have $\psi = \sin 2\theta$ and $\chi = 0$, as desired.

If we now differentiate ψ with respect to r and set $r = 1$, we get zero; Sage will do this for us as follows:

```
p = -1/12 * r^3/(r^2+1)^2 - 1/6* r*(r^2+3)/(r^2+1)^3 - 5*r/48
print(p.diff(r)(r=1))
```

This is not an accident; instead it is another sign that these calculations are in good order, since it can be shown that if k is a sufficiently smooth tangent vector, then k is in the kernel of $D^2E(X)$ if and only if $\psi_r = 0$ on the boundary.

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