

# Erdős Problem 633 about Triangle Tiling

Michael Beeson

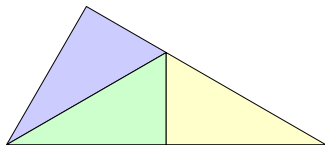
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Joint work with Miklós Laczkovich (Budapest)  
and Yan Zhang (SJSU).

## Definition

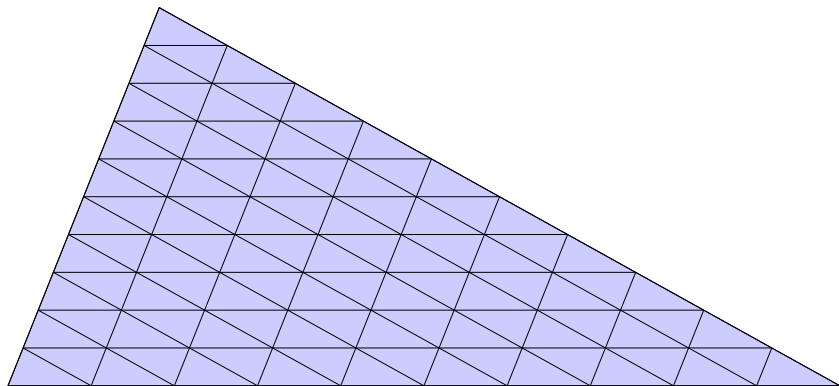
A triangle tiling is a way to write a triangle  $T$  as a union of smaller congruent triangles overlapping only at their boundaries.



The “tiles” have to be congruent, not just similar, and they don’t have to be similar to  $T$ .

## Quadratic tiling of any triangle

Any triangle can be cut into  $m^2$  tiles for any  $m$  by drawing  $m - 1$  lines parallel to each side:



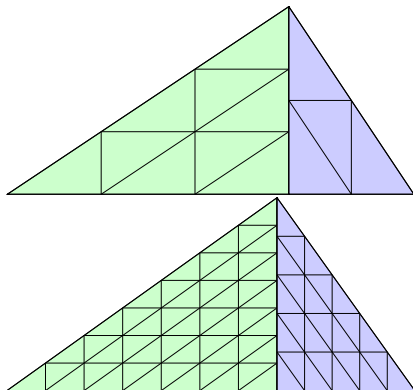
# The Erdős problem we solved

Classify those triangles which can only be tiled by square tilings.

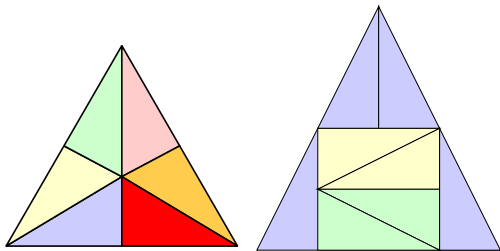
I will approach this problem like a botanist. We will go on a field trip to collect some samples.

# Biquadratic tilings with $N = e^2 + f^2$

Here we have  $N = 13 = 3^2 + 2^2$  and  $N = 74 = 5^2 + 7^2$ .

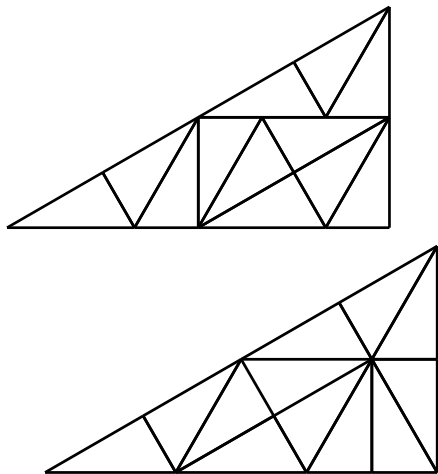


## A 6-tiling and an 8-tiling



## The $3m^2$ tilings

$N = 3m^2$ , and both the tile and the tiled triangle are 30-60-90 triangles. Here is the case  $m = 2$  and  $N = 12$ :



These are *reptiles*: the tile is similar to the tiled triangle.

# Snover's theorem on reptiles

All the tilings so far are *reptiles*: the tile is similar to the tiled triangle. Snover classified reptiles in 1991:

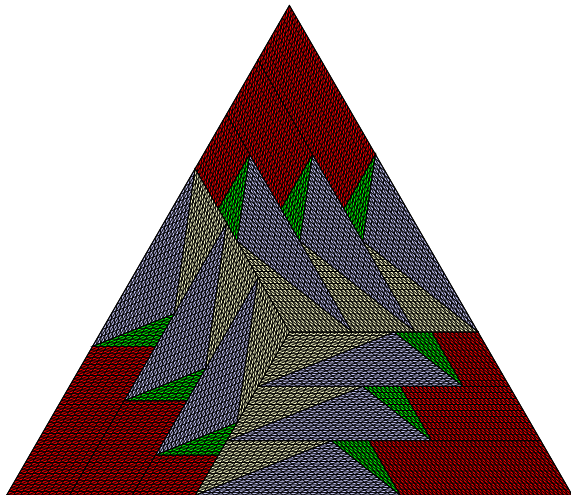
## Theorem (Snover *et. al.*)

*Suppose  $ABC$  is  $N$ -tiled by tile  $T$  similar to  $ABC$ . If  $N$  is not a square, then  $T$  and  $ABC$  are right triangles. Then either*

- (i)  $N$  is three times a square and  $T$  is a 30-60-90 triangle, or*
- (ii)  $N$  is a sum of two squares  $e^2 + f^2$ , the right angle of  $ABC$  is split by the tiling, and the acute angles of  $ABC$  have rational tangents  $e/f$  and  $f/e$ ,*

*and these two alternatives are mutually exclusive.*

$N = 10395$  with tile  $(3, 5, 7)$ , found about 2012



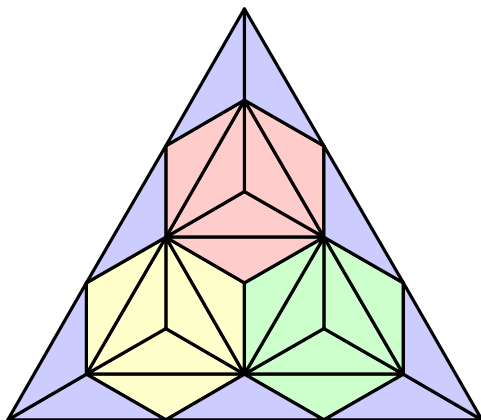
$N$  seemed rather large at the time. Can it be smaller?

As “botanists” we need some classifying properties of any triangle:

- ▶ *Commensurable angles*: the angles are all rational multiples of  $\pi$ .
- ▶ *Commensurable sides*: the ratios of the sides are rational, so up to similarity the triangle has integer sides.

Let's review the tilings so far to see which tiles and triangles have which properties.

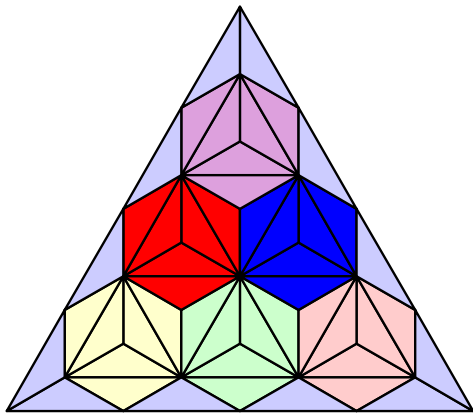
## A 27-tiling



October 12, 2008: I found a family of  $3m^2$  tilings, built from hexagons.

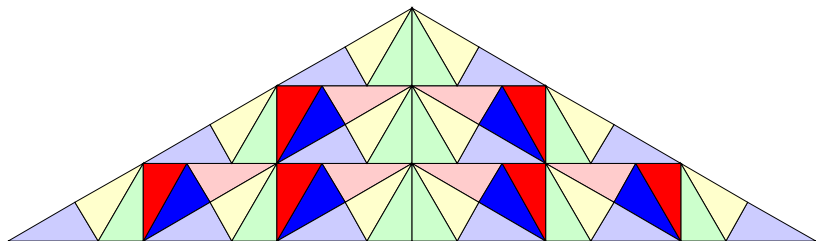
In 2012 I found that perhaps this one was known to Major MacMahon long ago (1921).

$3m^2$  tiling for  $m = 4$ ,  $N = 48$



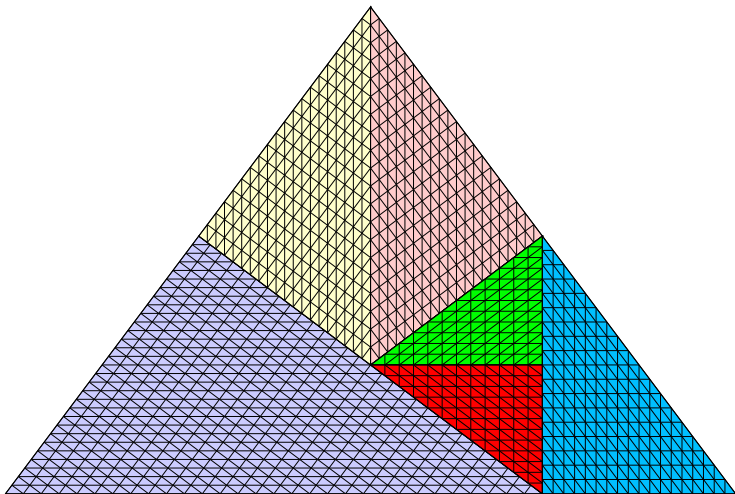
This tiling is made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides.

A 54-tiling;  $N/2$  is three times a square. Tile is 30-60-90.



$$N = 2312, N/2 = 34^2, (a, b, c) = (3, 4, 5)$$

Here  $2\alpha + 2\beta = \pi$ , that is,  $\gamma = \frac{\pi}{2}$ , and  $ABC$  is isosceles.



## The classification problem, a start

Classify the triples  $(ABC, T, N)$  such that triangle  $ABC$  can be  $N$ -tiled by tile  $T$ .

$T$  has sides  $(a, b, c)$  and angle  $(\alpha, \beta, \gamma)$ .

Triangle tilings are classified this way:

- ▶  $ABC$  is similar to the tile
- ▶ commensurable angles (all the angles rational multiples of  $2\pi$ )
- ▶ non-commensurable angles

Let's classify the examples we already saw.

## When the angles of $ABC$ are all rational multiples of $\pi$

Then the coordinates of all the vertices in the tiling belong to some cyclotomic field  $\mathbb{Q}(\zeta)$ , with  $\zeta = e^{2\pi i/k}$ .

That field has well-known automorphisms. Those automorphisms preserve lines and angles (but not lengths). So they take triangles into similar triangles, but not necessarily congruent triangles.

Therefore a tiling of  $ABC$  goes into a tiling of some other triangle, but only by *similar* triangles, not congruent ones. Then one works with those equations and eventually determines the possible tilings (after a lot of calculations).

Laczkovich carried out this work in 1990 and 1995 and eliminated more cases in 2012!

Table: Laczkovich's 2012 final answer

These are all the non-reptile tilings of  $T$  by  $(\alpha, \beta, \gamma)$  with commensurable angles:

$T$	the tile
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$
<i>not possible</i>	$(\frac{\pi}{3}, \frac{\pi}{12}, \frac{7\pi}{12})$
<i>not possible</i>	$(\frac{\pi}{3}, \frac{\pi}{30}, \frac{19\pi}{30})$
<i>not possible</i>	$(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30})$

# Our field trip

In the “botanist” metaphor:

- ▶ Commensurable angles is the “palace garden”
- ▶ Incommensurable angles is the “rain forest”

## Triangles with a 120 degree angle

Law of cosines tells us

$$c^2 = a^2 + b^2 - 2ab \cos(2\pi/3) = a^2 + b^2 - 2ab\left(-\frac{1}{2}\right)$$

$$c^2 = a^2 + b^2 + ab.$$

For example (3, 5, 7).

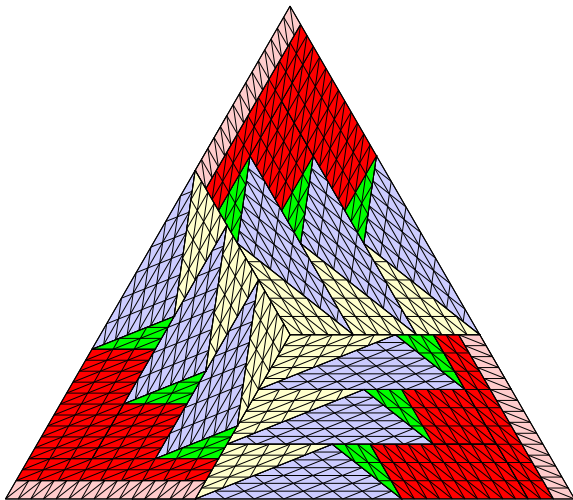
The angles are *incommensurable*:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \alpha \\ \cos \alpha &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{5^2 + 7^2 - 3^2}{2 \cdot 5 \cdot 7} \\ &= \frac{65}{70} = \frac{13}{14} \end{aligned}$$

But if  $\theta$  is a rational multiple of  $\pi$  and  $\cos \theta$  is rational only if it is 0 or  $\pm\frac{1}{2}$  or  $\pm 1$ , a fact known as “Niven’s theorem”. You probably saw it in Number Theory 101.

# A 1215-tiling of an equilateral triangle by $(3, 5, 7)$

In 2024, Bryce Herdt emailed me instructions to make this:



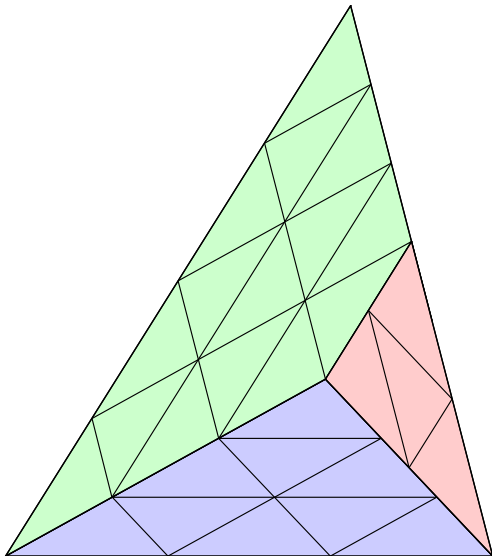
## The case $3\alpha + 2\beta = \pi$

In Fall 2011, I took up the case  $3\alpha + 2\beta = \pi$ . The first shape of  $ABC$  that I considered was  $(2\alpha, \beta, \alpha + \beta)$ .

- ▶ I tried hard to prove no such tilings existed.
- ▶ I found some necessary conditions so I knew that  $N$  had to be at least 28.
- ▶ I also found equations that told me what shape the tile had to be, namely  $(2, 3, 4)$ .
- ▶ Eventually I wrote a computer program which was supposed to show there was no 28-tiling.
- ▶ On October 11, 2011, with the help of that program, I found that I had been wrong: there do exist such tilings.

This was the first of many surprises in this subject.

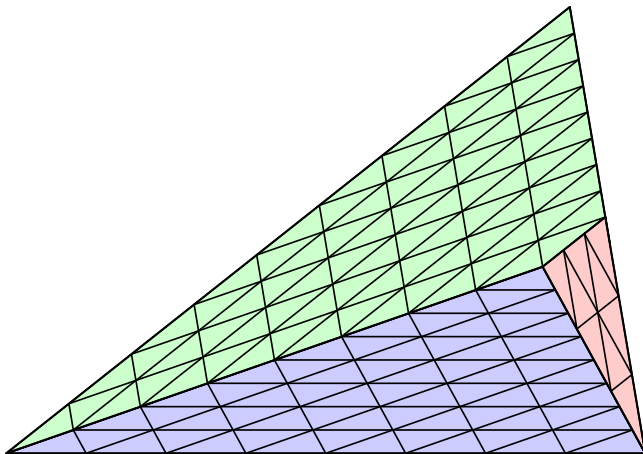
# A 28-tiling



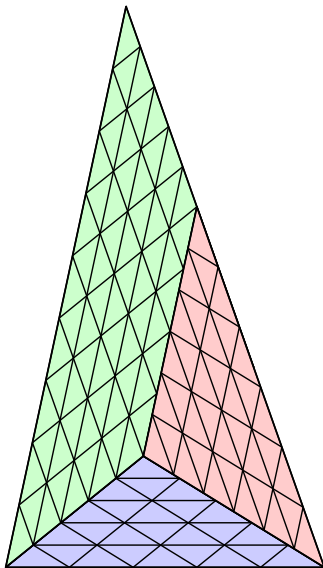
# The triquadratic tilings

The 28-tiling is just the first member of a family of *triquadratic tilings*. These are called that because they are almost composed of three quadratic tilings, but one of the three is missing a few tiles.

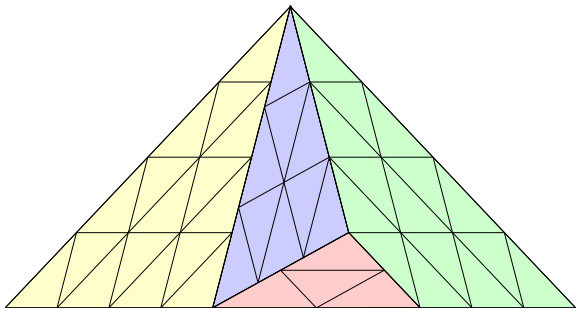
# A triquadratic tiling with $N = 153$



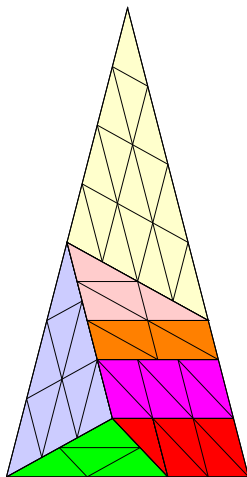
# A triquadratic tiling with $N = 126$



A 44-tiling with  $3\alpha + 2\beta = \pi$

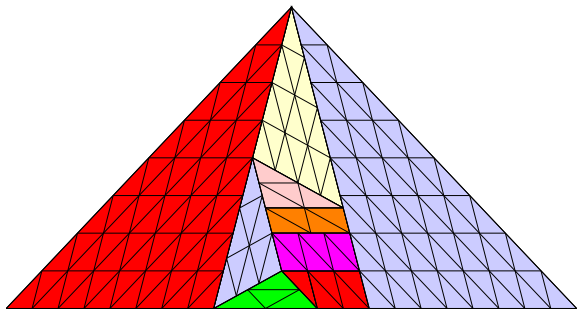


$$N = 48, 3\alpha + 2\beta = \pi, (a, b, c) = (2, 3, 4)$$



Here  $ABC$  is isosceles with base angles  $\alpha + \beta$ .

## A 176-tiling with $3\alpha + 2\beta = \pi$

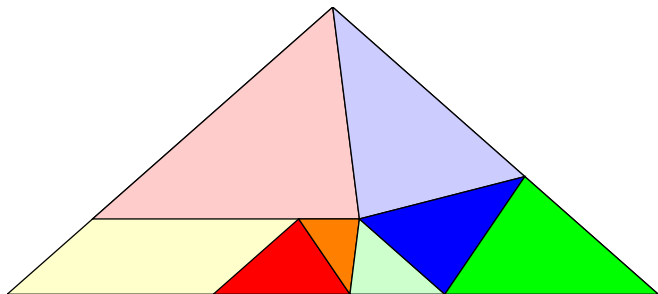


We just pasted on two quadratic “wings” to the previous isosceles tiling. Now the base angles are  $\beta$  and the vertex is  $3\alpha$ .

## Dissection of $ABC$ into triangles similar to $(4, 5, 6)$

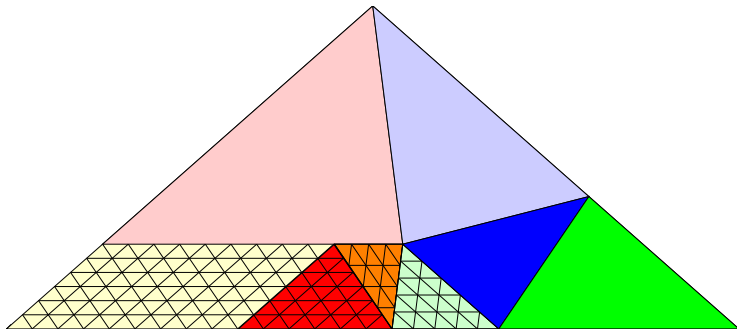
You could cut these triangles all into smaller *congruent* triangles, but you would need 6028020 of them.

Here  $3\alpha + \beta = \pi$  and  $ABC$  is isosceles. That is,  $\gamma = 2\alpha$ .



## We look for a tiling starting like this

Remember the tile is (4, 5, 6). Across every color border the lengths must match.



$$\begin{array}{ll} \text{red} & = 6 \\ \text{orange} & = 4 \text{red}/6 \\ \text{lightgreen} & = 5 \text{orange}/4 \\ \text{blue} & = 6 \text{lightgreen}/4 \\ \text{green} & = 5 \text{blue}/4 \\ \text{lightblue} & = 6 \text{blue}/4 \end{array} \quad \begin{array}{ll} \text{pink} & = 5 \text{lightblue}/4 \\ m & = (5 \text{pink} - 4 \text{orange})/6 \\ n & = (5 \text{red})/5 \\ \text{total} & = \text{red}^2 + \text{orange}^2 + \text{blue}^2 \\ & + \text{green}^2 + \text{lightblue}^2 \\ & + \text{pink}^2 + 2mn. \end{array}$$

Solving these equations, starting with  $red = 6$ , we get these answers (in the order of variables listed above):

$$6 \quad 4 \quad 5 \quad \frac{15}{2} \quad \frac{75}{8} \quad \frac{45}{4} \quad \frac{225}{16} \quad \frac{869}{96} \quad 6 \quad .$$

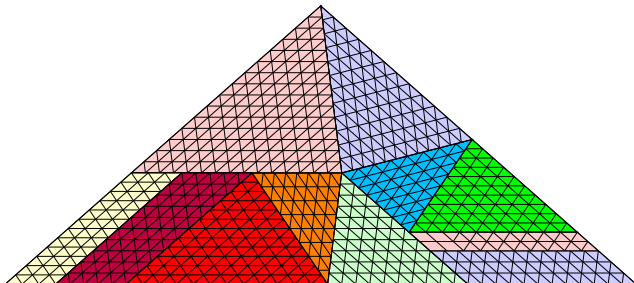
and clearing denominators, the final number of tiles required is

$$N = 6028020.$$

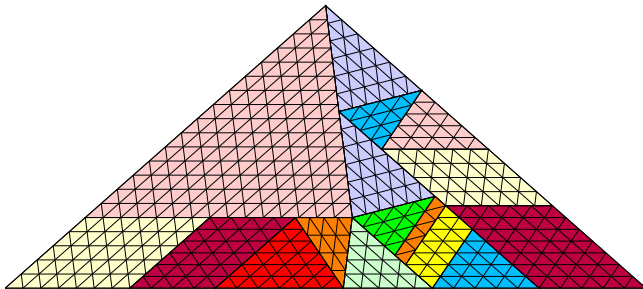
You might be able to draw that tiling on the side of a skyscraper; but not on a computer screen or piece of paper.

# Bryce Herdt to the rescue with an 1125-tiling

Until 2024, I did not know how to make a tiling of an isosceles triangle with base angles  $\alpha$  and vertex angle  $\alpha + \beta$  with few enough tiles to draw the picture. But then:



Then Herdt outdid himself with a 720-tiling



## What we found in the rain forest

- ▶ tilings with  $3\alpha + \beta = \pi$ , i.e.,  $\gamma = 2\alpha$  (isosceles  $ABC$ )
- ▶ tilings with  $3\alpha + 2\beta = \pi$  (several shapes of  $ABC$ ) Group 1
- ▶ tilings with  $3\alpha + 3\beta = \pi$ , i.e.,  $\gamma = 2\pi/3$ ,  $\alpha + \beta = \pi/3$  Group 2

Why? Why no tilings with, say,  $4\alpha + 7\beta = \pi$ ?

## Why not $4\alpha + 7\beta = \pi$ ?

Classify the vertices of the tiling by how many of angles  $\alpha, \beta, \gamma$  occur. For example a *center* has three  $\gamma$  angles, type  $(0, 0, 3)$ .

- ▶ No  $\gamma$  at a vertex of  $ABC$  (except for reptilings)
- ▶ in total  $N$  each of  $\alpha, \beta, \gamma$
- ▶ Therefore some vertex has fewer  $\alpha$  than  $\gamma$
- ▶ The sum of the angles at every vertex is  $2\pi$

And then comes algebra and trig... *That is too many equations for the angles to be incommensurable.*

## Assuming incommensurable angles

Analyzed by Miklos Laczkovich 1995.

Any tiling falls in one of these cases:

$ABC$	the tile
equilateral	$\alpha = \pi/3$
equilateral	$\gamma = 2\pi/3$
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
$(\alpha, \alpha, \alpha + \beta)$	$\gamma = 2\alpha$
several possible shapes	$3\alpha + 2\beta = \pi$
several possible shapes	$\gamma = 2\pi/3$

But of course most  $\alpha, \beta$  won't correspond to a tiling.

## Tiling existence theorem (Laczkovich 1995)

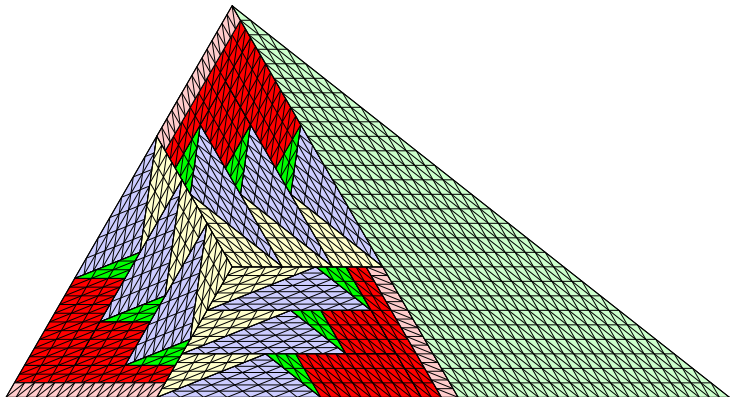
The conditions for the sides of the tile to be commensurable are

- ▶  $\sin(\alpha/2) \in \mathbb{Q}$  for  $3\alpha + 2\beta = \pi$  (Group 1)
- ▶  $\sqrt{3} \tan(\alpha/2) \in \mathbb{Q}$  for  $\gamma = 2\pi/3$  (Group 2)

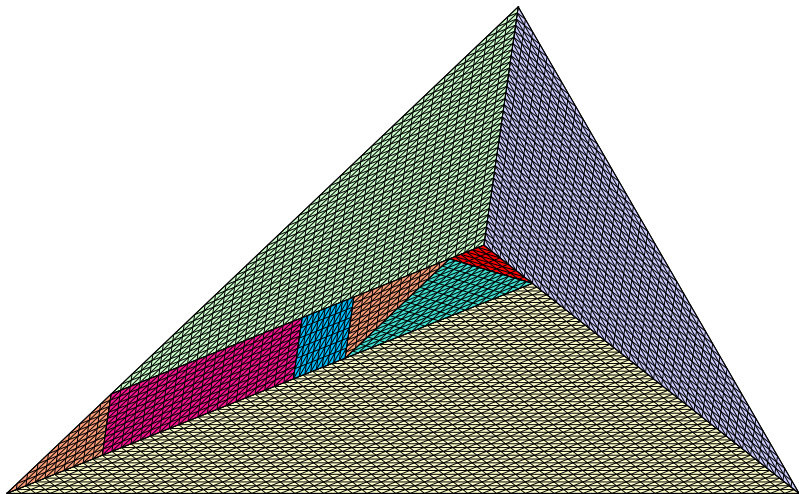
That is just an exercise in trigonometry and cyclotomic fields.

And under those conditions, dissections into similar triangles exist for all possible shapes of  $T$ , and hence,  $N$ -tilings for large enough  $N$ , by using smaller and smaller tiles. Laczkovich offers hand-drawn diagrams, and everyone seems to believe they will work for all choices of  $\alpha$ ; at least we drew them with computer precision.

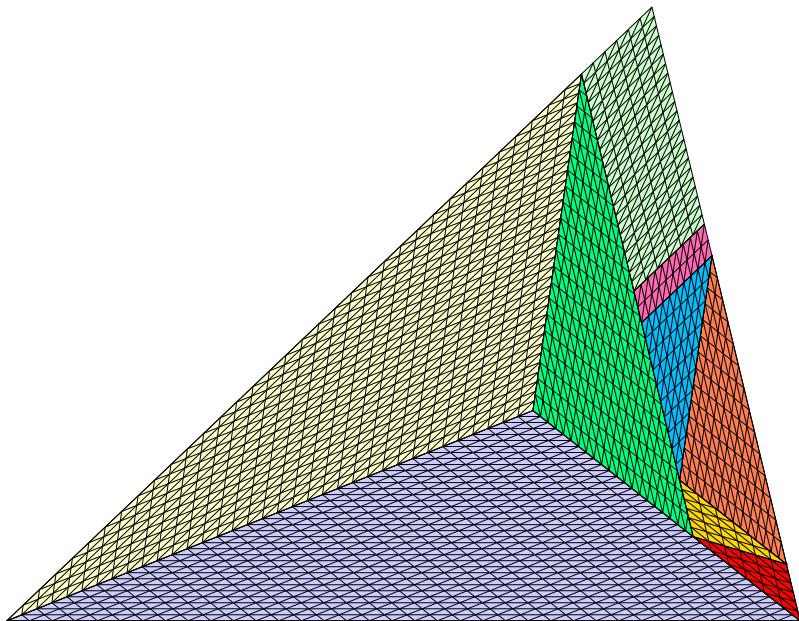
A 1944-tiling by  $(5, 3, 7)$ . Angles of  $T$  are  
 $(\alpha, \beta + \pi/3, \pi/3) = (\alpha, \alpha + 2\beta, \alpha + \beta)$



7007 tiles, angles  $(2\alpha, 2\beta, \alpha + \beta)$ , tile  $(3, 5, 7)$



3575 tiles, angles  $(2\alpha, 2\beta, \alpha + \beta)$ , tile  $(3, 5, 7)$



## Commensurable sides?

We saw that if the sides of the tile are commensurable and  $ABC$  has a suitable shape (its angles are combinations of  $\alpha$  and  $\beta$ ), then there is a tiling (for large enough  $N$ ). But that leaves open the converse: if there is a tiling, and the angles are incommensurable, are the sides necessarily commensurable?

- ▶ Laczkovich 2012 proved it for certain of the possible shape templates, but not all.
- ▶ I proved it for some other cases, including all the  $3\alpha + 2\beta$  cases and for isosceles tilings, and for some equilateral tilings.
- ▶ When I spoke here in 2019, some cases were still open, especially with  $\gamma = 2\pi/3$ .

## Theorem

Let triangle  $ABC$  be tiled by a tile  $R$  such that

- ▶  $R$  is not similar to  $ABC$ ;
- ▶  $R$  is not a right triangle;
- ▶  $R$  does not have commensurable angles.

Then  $R$  must have commensurable sides.

In short: **Incommensurable angles usually implies commensurable sides.**

## The area equation

$N$  times the area of the tile = area of  $ABC$ .

Twice the area of a triangle is the product of two sides times the sine of the included angle. Hence, the area equation, where  $ABC$  has side lengths  $(X, Y, Z)$  opposite angles  $(A, B, C)$ :

$$XZ \sin B = Nac \sin \beta$$

If, for example, the angle at  $B$  is  $\beta$  then

$$XZ = Nac$$

We get a different area equation for each shape of  $ABC$ .

## Our plan to solve Erdős 633

- ▶ Express  $N$  as a function of a rational parameter  $s$  depending on the angles of the triangle  $T$ . For example,  $s = \sin(\alpha/2)$ .
- ▶ Then  $N = Z^2$  is a Diophantine equation in  $s$  and  $Z$ .
- ▶ If it has no solutions then there are only square tilings for that triangle.
- ▶ Since there *are* tilings, there are non-square tilings.
- ▶ Find the equation for each possible family of tilings.
- ▶ Show those equations have no rational solutions.

## Let's work an example

Let the big triangle have incommensurable angles  $(A, B, \pi/3)$  with

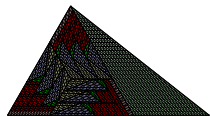
$$\sqrt{3} \tan(A/2) \in \mathbb{Q}.$$

Then

1. it has a tiling into  $(\alpha, \beta, \gamma)$  with  $\alpha = A$  and  $\gamma = 2\pi/3$ .
2. Yet to prove: the number of tiles is not a square.

We have seen that

- ▶ the sides are commensurable

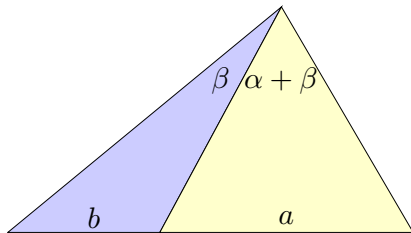


- ▶ the tiling exists. For example we saw
- ▶ the angles of the tiled triangle are  $(\alpha, \alpha + 2\beta, \alpha + \beta)$ .
- ▶ In the example  $N = 1944 = 2^3 \cdot 3^5$  is not a square. But was that accidental?

## A trigonometric identity we'll need

Suppose  $\alpha + \beta = \frac{\pi}{3}$ . Then

$$\sin(\alpha + 2\beta) = \sin \alpha + \sin \beta.$$



$$\frac{\sin(\alpha + 2\beta)}{a + b} = \frac{\sin \alpha}{a} \quad \text{by the law of sines}$$

$$\begin{aligned} \sin(\alpha + 2\beta) &= a \frac{\sin \alpha}{a} + b \frac{\sin \alpha}{a} \\ &= \sin \alpha + \sin \beta \end{aligned}$$

## Use the area equation

As we just proved

$$\sin(\alpha + 2\beta) = \sin \alpha + \sin \beta.$$

Recall the angles of  $T$  are  $(\alpha, \alpha + 2\beta, \alpha + \beta)$ .

Then by the law of sines, the sides of  $T$ , facing angles  $(\alpha, \alpha + 2\beta, \alpha + \beta)$ , are in ratio of

$$(a, a + b, c).$$

The sides of the triangle  $T$  must be

$$(ma, m(a + b), mc) \quad \text{for some integer } m$$

Then the area equation is

$$ma \cdot m(a + b) \sin(\pi/3) = N ab \sin(2\pi/3)$$

$$m^2(a + b) = Nb \quad \text{since } \sin(\pi/3) = \sin(2\pi/3)$$

$$b(a + b)m^2 = Nb^2$$

For  $N$  to be a square,  $b(a + b)m^2$  and hence  $b(a + b)$  also must be squares. But  $c^2 = a^2 + b^2 + ab$  because the tile has a  $2\pi/3$  angle.

## Now for the number theory

### Theorem

*Suppose  $a$  and  $b$  are coprime positive integers, then  $a^2 + b^2 + ab$  and  $b(a + b)$  cannot both be squares.*

*Proof.* Put  $b = m^2$  and  $(a + b) = n^2$  and  $c^2 = a^2 + b^2 + ab$ . After a little algebra we have

$$c^2 = n^4 - m^2n^2 + m^4.$$

We have to show that equation has no positive integer solutions. It turns out number theorists know things like this. Pocklington published a proof in 1914! You can also look it up in the comprehensive textbook by Cohen. The answer is that  $(m, n, c) = (t, t, t^2)$  for some  $t$ . Then  $a = b = 0$ , since  $b = m^2 = t^2$  and  $a + b = n^2 = t^2$ .

So we're done, for this particular shape of  $T$ .

## The modern way: elliptic curves

We'll work that example again as if Pocklington hadn't done it in 1914. We'll reduce it to an elliptic curve by substitution. First, put

$$t = n/m \quad \text{and} \quad s = c/m^2.$$

Then the equation becomes

$$s^2 = t^4 - t^2 + 1.$$

That's a special case of

$$s^2 = t^4 + at^2 + b.$$

To transform that to Weierstrass form, put

$$x = 2t^2 - 2s + a \quad \text{and} \quad y = 2tx$$

and we get

$$y^2 = x^3 - 2ax^2 + (a^2 - 4b)x$$

which is an elliptic curve.

## Analyze the elliptic curve

We had in general

$$y^2 = x^3 - 2ax^2 + (a^2 - 4b)x$$

In our case  $a = -1$  and  $b = 1$ , so our curve is

$$y^2 = x^3 + 2x^2 - 3x.$$

We want to analyze this curve as follows:

- ▶ the group  $\Gamma$  of rational points has rank 0
- ▶ the points of finite order are

$$(0, 0), (1, 0), (-3, 0), (-1, 2), (-1, -2), (3, 6), (3, -6)$$

and the point at infinity

- ▶ Those correspond to  $(t, s) = (0, \pm 1), (\pm 1, \pm 1)$
- ▶  $(0, \pm 1)$  would imply  $m = 0$ , which would mean  $a = 0$ .
- ▶  $(\pm 1, \pm 1)$  would imply  $m = n$ , which would mean  $b = 0$ .

So we're done again, once we find the rank and points of finite order.

## Find the rank and finite-order points

You could analyze an elliptic curve from scratch as explained in textbooks like Silverman-Tate. But:

The modern way to find out about an elliptic curve is to look it up in the Cremona database. That can be done

- ▶ by looking it up on a webpage
- ▶ by using SageMath

## SageMath output

Elliptic Curve defined by  $y^2 = x^3 + 2x^2 - 3x$  over Rational Field

Conductor: 24

Minimal model: Elliptic Curve defined by

$y^2 = x^3 - x^2 - 4x + 4$  over Rational Field

Rank: 0

Torsion subgroup: Torsion Subgroup isomorphic to  $\mathbb{Z}/4 + \mathbb{Z}/2$

associated to the Elliptic Curve defined by

$y^2 = x^3 + 2x^2 - 3x$  over Rational Field

Torsion order: 8

Torsion points:

(-3 : 0 : 1)

(-1 : -2 : 1)

(-1 : 2 : 1)

(0 : 0 : 1)

(0 : 1 : 0)

(1 : 0 : 1)

(3 : -6 : 1)

(3 : 6 : 1)

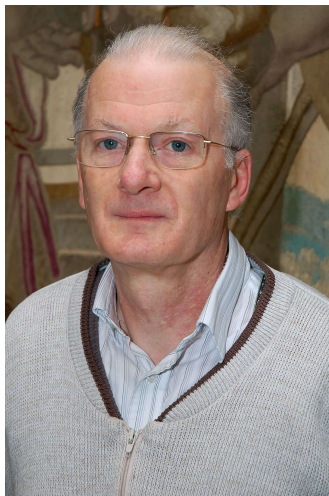
## The solution of Erdős 633

A triangle  $T$  admits a non-square tiling if and only if it satisfies one of the following conditions, where  $(A, B, C)$  are the angles of  $T$  in some order:

- (1)  $A = B$ , i.e.  $T$  is an isosceles triangle (including equilateral);
- (2)  $C = \pi/2$  and the legs of the right triangle  $T$  are in integer ratio  $M/K$ , where  $M^2 + K^2$  is not a square;
- (3)  $(A, B, C) = (\pi/6, \pi/2, \pi/3)$ ;
- (4)  $C = \pi/3$ , with  $\sqrt{3} \tan(A/2)$  rational;
- (5)  $B = 2A$ , with  $\sqrt{3} \tan(A/2)$  rational;
- (6)  $B = 2A$ , with  $\sin(A/2)$  rational;
- (7)  $C = A/2 + B$ , with  $2 \sin(A/4)$  rational, equal to  $M/K$ , where  $2K^2 - M^2$  is not a square.
- (8)  $C = 2A + B/2$ , with  $\sqrt{3} \tan(A/2)$  rational.

## Miklós Laczkovich

Miklós Laczkovich has written many papers on triangle tiling, starting in 1995 and continuing to the present day. He is a retired professor living in Budapest. He has also written uncountably many papers on other subjects. I have never met him in person.

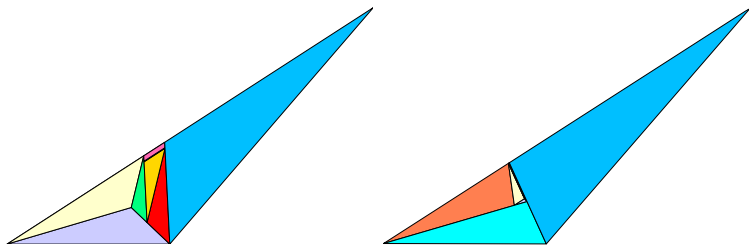


# Yan Zhang

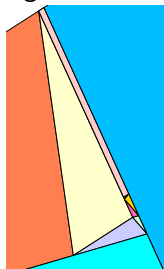
Yan wrote me in 2024. In 2026 we wrote two papers together, without ever meeting. After that, I took him out to dinner.



## Same triangle tiled by two different tiles



Left:  $3\alpha + 3\beta = \pi$ . Right:  $3\alpha + 2\beta = \pi$ . Below: details



The number of tiles in the second tiling exceeds  $10^{20}$ .